RNC WORKSHOP
ABSOLUTE MINIMIZERS IN CARNOT GROUPS

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Abstract. In this note, we extend the concepts of viscosity solutions and absolute minimizers to the setting of Carnot groups. In particular, the existence-uniqueness of infinite harmonic functions in the viscosity sense and the relationship between absolute minimizers and infinite harmonic functions are discussed. As a consequence, the uniqueness of absolute minimizers follows.

1. Carnot Groups

We begin by considering $\mathbb{R}^N$ for some integer $N > 2$ with a nilpotent stratification that decomposes $\mathbb{R}^N$ into

$$V_1 \oplus V_2 \oplus \cdots \oplus V_w$$

for appropriate vector subspaces V_i that satisfy the Lie bracket relation $[V_1, V_i] = V_{i+1}$ for $i = 1, 2, \ldots, w-1$. That is, V_1 generates the stratification. There is an orthonormal basis under some fixed Riemannian metric $\langle \cdot, \cdot \rangle$ denoted by

$$X_{11}, X_{12}, \ldots, X_{1n_1}, X_{21}, X_{22}, \ldots, X_{2n_2}, \ldots, , X_{w1}, X_{w2}, \ldots, X_{wn_w}$$

so that

$$V_i = \operatorname{span}\{X_{i1}, X_{i2}, \ldots, X_{in_i}\}.$$

Let g be the Lie Algebra generated by these vector fields. A vector in g can be associated with an element in the Carnot group G by identifying $\sum_{i=1}^{w} \sum_{j=1}^{n_i} x_{ij} X_{ij}$ with $(x_{11}, x_{12}, \ldots, x_{wn_w})$. This identification is used to define a group product on G via the Campbell-Baker-Hausdorff formula

$$p \cdot q = p + q + \frac{1}{2}[p, q] + R(p, q)$$

where $R(p, q)$ involves Lie brackets of order 2 and higher. Note that by skew-symmetry of the Lie bracket, a Carnot group is, in general, non-abelian. In particular, right and left multiplication are not equivalent.

In control theory, one is interested in equations of the form

$$x'(t) = \sum_{i=1}^{n_1} u_i(t) X_{1i}(x)$$

for L^1 functions u_i defined on some interval. (See, for example, [Be].) Since the vectors in V_1 do not necessarily span g, there is a loss of information in certain directions, creating a sub-Riemannian structure. (The role of Carnot groups to sub-Riemannian geometry is analogous to that of Euclidean spaces and Riemannian geometry. [Be]) In particular, only the vectors in V_1 are used to define the horizontal gradient of a

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function. That is, given a function $u : G \mapsto \mathbb{R}$, the horizontal gradient of u is given by

$$\nabla_0 u = (X_{11}u, X_{12}u, \ldots, X_{1n_1}u).$$

Here one can see that if the dimension of g is significantly larger than the dimension of V_1, there may be a great amount of missing information. Also, note that vector fields in V_i are actually ith order derivatives, which means the full gradient, which is defined using all the vectors in the basis for g, contains derivatives up to wth order. In particular, $\nabla_0 u$ is the projection of the full gradient onto V_1. We will also consider two second order derivatives, namely, the semi-horizontal gradient, defined by

$$\nabla_1 u = (X_{11}u, X_{12}u, \ldots, X_{1n_1}u, X_{21}u, \ldots, X_{2n_2}u)$$

and the symmetrized horizontal second derivative matrix, denoted by $(D^2u)^\star$, with entries

$$((D^2u)^\star)_{ij} = \frac{1}{2}(X_{1i}X_{1j}u + X_{1j}X_{1i}u)$$

for $i, j = 1, 2, \ldots, n_1$. Using these second order derivatives leads to a natural definition.

Definition 1. A function u is C^2_{sub} if $\nabla_1 u$ is continuous and $X_{1i}X_{1j}u$ is continuous for all i, j.

The set of C^2_{sub} functions is strictly larger than the set of Euclidean C^2 functions. For example, a quick calculation (or Theorem 4.10 in [Be]) shows that the function

$$f(x_{11}, x_{12}, \ldots, x_{wn_w}) = (x_{21})^{\frac{3}{2}}$$

is C^2_{sub} at the origin, but it is clearly not C^2 in the Euclidean sense at the origin.

Carnot groups are endowed with a natural metric based on horizontal curves, which are defined by $\gamma : \mathbb{R} \mapsto G$ with tangent vector $\gamma'(t)$ in V_1. Using these curves, the Carnot-Carathéodory distance is defined for the points p and q as follows:

$$d_C(p, q) = \inf_{\Gamma} \int_0^1 \langle \gamma'(t), \gamma'(t) \rangle^{\frac{1}{2}} dt$$

where the set Γ is the set of all horizontal curves γ such that $\gamma(0) = p$ and $\gamma(1) = q$. By Chow's theorem (See, for example, [Be].) any two points can be connected by a horizontal curve, which makes $d_C(p, q)$ a left-invariant metric on G.

In addition to the (non-smooth) Carnot-Carathéodory metric, there is a smooth gauge bi-Lipschitz equivalent to the metric. Given a point p with coordinates x_{ij} as above, the gauge is given by

$$\|p\|^{2(w!)} = \sum_{i=1}^{w} (\sum_{j=1}^{n_i} |x_{ij}|^2)^{\frac{w!}{i}}.$$

The gradation of g produces a natural family of dilations given by δ_s, for $s > 0$, so that

$$\delta_s(V_i) = s^i V_i$$

and this induces a family of dilations on G given by

$$\delta_s(p) = (sx_{11}, sx_{21}, \ldots, s^2x_{21}, s^2x_{22}, \ldots, s^w x_{wn_w}).$$

Both the metric and the gauge behave appropriately with respect to these dilations, namely,

$$\begin{aligned} d_C(\delta_s p, \delta_s q) &= s\, d_C(p,q) \\ \|\delta_s p\| &= s\|p\|. \end{aligned}$$

Two well-known examples of Carnot groups are Euclidean space and the Heisenberg group. The Euclidean space $\mathbb{R}^N$ is a Carnot group with $w = 1$ and with the vectors $X_{1,j} = e_j$ (the canonical vectors). In addition, the Carnot-Carathéodory metric and smooth gauge are both equivalent to the standard Euclidean metric and the dilations are the usual isotropic dilations. Having only one non-trivial Lie bracket, the Heisenberg group is a non-abelian Carnot group most closely resembling Euclidean space. We shall consider the first Heisenberg group $\mathbb{H}_1$, but the following can be extended in the natural way to higher dimensional Heisenberg groups. The Heisenberg algebra h_1 can be identified with $\mathbb{R}^3$ with coordinates (x, y, z) spanned by a basis consisting of vector fields X_1, X_2, and X_3 given at a point (x_0, y_0, z_0) by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} - \frac{y_0}{2}\frac{\partial}{\partial z}, \\ X_2 &= \frac{\partial}{\partial y} + \frac{x_0}{2}\frac{\partial}{\partial z}, \\ \text{and} \quad X_3 &= \frac{\partial}{\partial z}. \end{aligned}$$

Using these vector fields, the only non-trivial Lie bracket relationship is $[X_1, X_2] = X_3$ and most notably, all Lie brackets of length at least 2 are trivial. This shows that $\mathbb{H}_1$ is a Carnot group with $w = 2$ and

$$\begin{aligned} V_1 &= \operatorname{span}\{X_1, X_2\} \\ V_2 &= \operatorname{span}\{X_3\}. \end{aligned}$$

We can easily compute that the gauge is given by

$$\|p\| = ((x^2 + y^2)^2 + z^2)^{\frac{1}{4}}$$

and the dilations are

$$\delta_s p = (sx, sy, s^2 z).$$

For a further discussion on Carnot groups, the interested reader is directed to [BMT], [BT], [Be], [B], [DK], [F], [FS], [GN], [G], [H], [HH], [K], [Ka], [Ko], [LMS], [MS], [Mi], [Mo], [MoS], and [St].

2. Viscosity Solutions in Carnot Groups

Using the horizontal derivatives defined in the previous section, one can define partial differential equations on Carnot groups. In particular, it is natural to consider fully nonlinear degenerate elliptic equations of the form

$$F(p, u(p), \nabla_1 u(p), (D^2 u(p))^\star) = 0$$

where $F(p, w, \eta, X)$ is increasing in w and decreasing in the symmetric matrix X. Our main example of such an F is the negative infinite Laplacian, defined by

$$-\Delta_{0,\infty} u = -\langle (D^2 u(p))^\star \nabla_0 u(p), \nabla_0 u(p) \rangle.$$

(The negative sign is needed to have the function F decrease as the symmetric matrix X increases.) Also, recall that the subelliptic P-Laplacian is defined for $1 < P < \infty$ by

$$\Delta_{0,P} u = \sum_{i=1}^{2} X_{1i}(\|\nabla_0 u\|^{P-2} X_{1i} u)$$

so that formally,

$$\Delta_{0,\infty} = \lim_{P\to\infty} \Delta_{0,P}.$$

For results concerning the P-Laplacian on the Heisenberg group, see [B], [CDG] and [BMT].

Another nonlinear example is the sub-Riemannian Monge-Ampère equation given by

$$\det((D^2 u(p))^\star) = f(p)$$

for some function f. Notice that this differs from the Euclidean version only by the use of the symmetrized horizontal derivative matrix. For a catalog of examples in the Euclidean case, see [CIL].

Working with this class of equations, it is natural to extend the concept of solution from smooth functions to functions that are merely continuous. This is done via viscosity solutions. We begin with some definitions.

Definition 2. A continuous function $u : G \mapsto \mathbb{R}$ is a *viscosity subsolution* at p_0 to

$$F(p_0, u(p_0), \nabla_1 u(p_0), (D^2 u(p_0))^\star) = 0$$

if for every C^2_{sub} function $\phi : G \mapsto \mathbb{R}$ so that $u - \phi$ has a local maximum at p_0 we have

$$F(p_0, \phi(p_0), \nabla_1 \phi(p_0), (D^2 \phi(p))^\star) \leq 0.$$

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if for every C^2_{sub} function $\psi : G \mapsto \mathbb{R}$ so that $u - \psi$ has a local minimum at p_0 we have

$$F(p_0, \psi(p_0), \nabla_1 \psi(p_0), (D^2 \psi(p))^\star) \geq 0.$$

A continuous function $u : G \mapsto \mathbb{R}$ is a *viscosity solution* at p_0 to

$$F(p_0, u(p_0), \nabla_1 u(p_0), (D^2 u(p_0))^\star) = 0$$

if it is both a viscosity subsolution and a viscosity supersolution.

Note that the test functions ψ and ϕ may not exist, in which case the definition is trivially satisfied. Existence of these functions (or lack thereof) is related to regularity of the function u.

We are mainly concerned with viscosity infinite harmonic functions, that is, using the notation of the above definition, functions u that satisfy

$$\begin{aligned} -\langle (D^2\phi(p))^\star \nabla_0 \phi(p_0), \nabla_0 \phi(p_0)\rangle &\leq 0 \\ -\langle (D^2\psi(p))^\star \nabla_0 \psi(p_0), \nabla_0 \psi(p_0)\rangle &\geq 0. \end{aligned}$$

In order to study sub-Riemannian viscosity solutions, one needs subelliptic versions of the machinery in [CIL], which is also an excellent reference for Euclidean viscosity solutions. In addition, the reader may examine [BDM], [Cr1], [Je], and [JLM]. For sub-Riemannian viscosity solutions, the reader is directed to [B], [BM], [LMS], [M],

[MS], and [W]. The following theorem concerning infinite harmonic functions in Carnot groups is proved in [BM] and [W], independently. It is an extension of the Euclidean case found in [BDM] and [Je].

Theorem 1. *Let $\Omega \subset G$ be a domain and $v\colon \partial\Omega \mapsto \mathbb{R}$ be a continuous function. Then the Dirichlet problem*

$$\begin{cases} \Delta_{0,\infty} u = 0 & in \quad \Omega \\ u = v & on \quad \partial\Omega \end{cases}$$

has a unique viscosity solution u.

3. Absolute Minimizers

In his paper, McShane [MC] showed that a Lipschitz function defined only on the boundary of a domain can be extended to the interior of the domain so that the extension function is Lipschitz with the same Lipschitz constant as the original function. That is, McShane proved that given the function $f \in \mathrm{Lip}_{\partial\Omega}$, there is a function $u \in \mathrm{Lip}_{\Omega}$ so that

$$\begin{cases} u = f \quad \text{on} & \partial\Omega \\ \mathrm{Lip}_{\Omega}(u) \quad = & \mathrm{Lip}_{\partial\Omega}(f). \end{cases}$$

However, these extensions are not unique; there may even be an uncountable number of them. Motivated by questions of uniqueness, G. Aronsson [A] studied the concept of minimal Lipschitz extensions, called absolute minimizers. In a Carnot group G, we define an absolute minimizer by the following.

Definition 3. A function $u \in \mathrm{Lip}(\overline{\Omega})$ is an absolute minimizer if for every $V \subset \Omega$ and $h \in \mathrm{Lip}(V)$, such that $u = h$ on ∂V, then

$$||\nabla_0 u||_{L^\infty(V)} \leq ||\nabla_0 h||_{L^\infty(V)}.$$

For a thorough discussion of absolute minimizers, the reader is directed to the article by Juutinen [Ju2], which appears in this volume. In addition, the reader may consult [A], [BJW], [BC], [Cr2], [CEG], [Ju1], and [MC].

It has been shown independently in [BC] and [W] that absolute minimizers in Carnot groups are viscosity infinite harmonic functions as given by the following theorem.

Theorem 2. *Let $\Omega \subset G$ be a domain and let $u : \Omega \mapsto \mathbb{R}$ be a function so that $u \in \mathrm{Lip}(\Omega)$. Then if u is an absolute minimizer, then u is viscosity infinite harmonic in Ω.*

Combining this result with the previous section, we have the following corollary. ([BM] and [W], independently.)

Corollary 3. *Let $\Omega \subset G$ be a domain and let $v\colon \partial\Omega \mapsto \mathbb{R}$ be Lipschitz on $\partial\Omega$. Then there is a unique absolute minimizer equal to v on $\partial\Omega$.*

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