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PROPERTIES OF INFINITE HARMONIC FUNCTIONS ON GRUSHIN-TYPE SPACES

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ABSTRACT. In this paper, we examine potential-theoretic and geometric properties of viscosity infinite harmonic functions in Grushin-type spaces, which are sub-Riemannian spaces lacking a group structure. In particular, we prove such functions enjoy comparison with Grushin cones. As a consequence, the distance function is viscosity infinite superharmonic, but we show that it is not necessarily viscosity infinite subharmonic and give geometric conditions when it is.

1. Introduction. The goal of this paper is to examine viscosity infinite harmonic functions in Grushin-type spaces from both the potential-theoretic and the geometric viewpoints. Motivated by the author's result in [**5**] that C^1_{sub} absolute minimizers are viscosity infinite harmonic (see Sections 3 and 4 for relevant definitions) and its improvement by Wang [**15**], who relaxes the regularity, we wish to establish the potential-theoretic properties of viscosity infinite harmonic functions. In particular, we wish to prove the following main theorem:

Main theorem. *Given a domain Ω and a function u, the following are equivalent.*

(1) *u is an absolute minimizer.*

(2) *u is viscosity infinite harmonic.*

(3) *u is potential harmonic.*

(4) *u enjoys comparison with Grushin cones.*

In addition, the corresponding "one-sided" statements hold. Namely, the following are equivalent.

(I) *u is an absolute sub (super)-minimizer.*

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(II) *u is a viscosity infinite sub (super)-harmonic.*

(III) *u is a potential infinite sub (super)-harmonic.*

(IV) *u enjoys comparison with Grushin cones from above (below).*

Focusing on the cone characterization in this theorem, we will then examine the geometry of the cones. Grushin-type spaces possess a rich geometry that has recently been explored in such papers as [**7, 8**]. This geometry, which at some points is Riemannian and some sub-Riemannian, provides a contrast to Euclidean spaces. Even though the Grushin distance function is viscosity infinite superharmonic, it need not be viscosity infinite subharmonic. This is unlike the Euclidean case, where the distance function is viscosity infinite superharmonic and subharmonic. We will give geometric conditions for when the distance function is viscosity infinite subharmonic.

The paper is divided up in the following manner. In Section 2, we provide the needed background properties of Grushin-type spaces. In Section 3, we show that viscosity infinite harmonic functions are unique. In Section 4, we give some potential-theoretic consequences of the existence-uniqueness. At this point, the first three characterizations in the Main theorem are proved. In Section 5, we define Grushin cones and give some of their basic properties as well as complete the proof of the Main theorem. In Section 6, we delve into the Harnack inequality and some of its consequences, which include Section 7, where we focus on the relationship between Grushin cones and the Grushin geometry. Section 8 shows how comparison with cones produces regularity results.

2. Grushin-type spaces. We begin by recalling the main properties of a Grushin-type space. For a more thorough discussion, the interested reader is referred to [**2, 5**] and the references therein.

We consider $\mathbf{R}^n$ with coordinates $(x_1, x_2, \dots, x_n)$ and the vector fields

$$X_i = \rho_i(x_1, x_2, \dots, x_{i-1}) \frac{\partial}{\partial x_i}$$

for $i = 2, 3, \dots, n$ where $\rho_i(x_1, x_2, \dots, x_{i-1})$ is a (possibly constant) polynomial. We decree that $\rho_1 \equiv 1$ so that

$$X_1 = \frac{\partial}{\partial x_1}.$$

A quick calculation shows that when $i < j$, the Lie bracket is given by

$$X_{ij} \equiv [X_i, X_j] = \rho_i(x_1, x_2, \dots, x_{i-1}) \frac{\partial \rho_j(x_1, x_2, \dots, x_{j-1})}{\partial x_i} \frac{\partial}{\partial x_j}. \tag{2.1}$$

Because the ρ_is are polynomials, Hörmander's condition is satisfied by these vector fields. Endowing $\mathbf{R}^n$ with an inner product (singular where the polynomials vanish) so that the X_is are orthonormal produces a manifold that we shall call g_n. This is the tangent space to a generalized Grushin-type space G_n. Points in G_n will also be denoted by $p = (x_1, x_2, \dots, x_n)$ with a fixed point denoted $p_0 = (x_1^0, x_2^0, \dots, x_n^0)$. In addition, we write $p - p_0$ for $(x_1 - x_1^0, x_2 - x_2^0, \dots, x_n - x_n^0)$ and abuse notation to write evaluation of the polynomials ρ_i at a point p_0 by $\rho_i(p_0)$.

The Carnot-Carathéodory distance is the natural metric and is defined for the points p and q as follows:

$$d_C(p, q) = \inf_{\Gamma} \int_0^1 \|\gamma'(t)\| \, dt$$

where the set Γ is the set of all curves γ such that $\gamma(0) = p$, $\gamma(1) = q$ and $\gamma'(t)$ is in span $\{\{X_i(\gamma(t))\}_{i=1}^n\}$. By Chow's theorem, see for example [**2**], any two points can be connected by such a curve. We are thus able to define a ball centered at p_0 with radius r, denoted $B(p_0, r)$, and Lipschitz functions in the obvious manner using this metric.

The Carnot-Carathéodory metric behaves differently when the polynomials $\rho_i(x_1, x_2, \dots, x_{i-1})$ vanish. Fixing a point p_0, consider the n-tuple $r_{p_0} = (r_{p_0}^1, r_{p_0}^2, \dots, r_{p_0}^n)$ where $r_{p_0}^i$ is the minimal length of the Lie bracket iteration required to produce

$$\Big[X_{j_1}, \Big[X_{j_2}, \Big[\cdots \Big[X_{j_{r_{p_0}^i}}, X_i\Big] \cdots\Big](p_0) \neq 0.$$

Clearly,

$$\rho_i(p_0) \neq 0 \longleftrightarrow r_{p_0}^i = 0.$$

Using [**2**, Theorem 7.34] we obtain the local estimate at p_0

$$d_C(p_0, p) \sim \sum_{i=1}^n |x_i - x_i^0|^{1/(1+r_{p_0}^i)}. \tag{2.2}$$

Given a smooth function f on G_n, we define the horizontal gradient of f as

$$\nabla_0 f(p) = (X_1 f(p), X_2 f(p), \dots , X_n f(p))$$

and the symmetrized second order (horizontal) derivative matrix by

$$((D^2 f(p))^\star)_{ij} = \frac{1}{2}(X_i X_j f(p) + X_j X_i f(p))$$

for $i, j = 1, 2, \dots n$. Using these derivatives, we may define the Sobolev spaces $W^{1,P}$ and $W^{1,P}_{\text{loc}}$ for $1 < P \leq \infty$ in the obvious way. We also have the following definition:

Definition 1. The function $f : G_n \to \mathbf{R}$ is said to be C^1_{sub} if $X_i f$ is continuous for all $i = 1, 2, \dots , n$. Similarly, the function f is C^2_{sub} if $X_i X_j f(p)$ is continuous for all $i, j = 1, 2, \dots , n$.

3. Viscosity infinite harmonic functions. Having constructed Grushin jets in [**5**], we recall their key properties. Following standard notation, we denote the Grushin superjet by $J^{2,+}$ and the subjet by $J^{2,-}$. We have the following proposition summarizing the results in [**5**].

Proposition 3.1. *Let $u : G_n \to \mathbf{R}$, and let $\mathcal{A}_{p_0}$ be the set of C^2_{sub} functions such that*

$$\phi \in \mathcal{A}_{p_0} \longleftrightarrow u - \phi \quad \text{has a local max at } p_0.$$

Then we have the characterization of the Grushin superjets by

$$J^{2,+}u(p_0) = \{(\nabla_0 \phi(p_0), (D^2 \phi(p_0))^\star) : \phi \in \mathcal{A}_{p_0}\}$$

and the characterization of the Grushin subjets by

$$J^{2,-}u(p_0) = -J^{2,+}(-u)(p_0).$$

We also note that we define the closure of the superjet, denoted $\overline{J}^{2,+}u(p_0)$, in the usual way [**9**].

The following twisting lemma ([**5**]) allows the Euclidean jets to be twisted into Grushin jets. This is a key connection between the Euclidean tools and Grushin-type spaces.

Lemma 3.2. *Let the points $p, p_0 \in \mathbf{R}^n$ be denoted by $p = (x_1, x_2, \ldots, x_n)$ and by $p_0 = (x_1^0, x_2^0, \ldots, x_n^0)$. Let $\eta \in \mathbf{R}^n$ and X be an $n \times n$ symmetric matrix such that $(\eta, X) \in \overline{J}^{2,+}_{\text{eucl}} u(p_0)$. Then $(\tilde{\eta}, Y) \in \overline{J}^{2,+} u(p_0)$ where the vector $\tilde{\eta}$ is defined by*

$$\tilde{\eta} = \sum_{i=1}^{n} \rho_i(p_0)\eta_i X_i$$

and the symmetric matrix Y is defined by

$$Y_{ij} = \begin{cases} \rho_i(p_0)\rho_j(p_0)X_{ij} + (1/2)(\partial\rho_j/\partial x_i)(p_0)\rho_i(p_0)\eta_j & i \leq j \\ Y_{ji} & i > j. \end{cases}$$

Using these jets, we have the following definition.

Definition 2. A lower semi-continuous function v is *viscosity infinite superharmonic* in the domain Ω if whenever $(\eta, \mathcal{Y}) \in \overline{J}^{2,-} v(p_0)$ with $p_0 \in \Omega$ we have

$$-\langle \mathcal{Y}\eta, \eta \rangle \geq 0.$$

An upper semi-continuous function u is a *viscosity infinite subharmonic* if whenever $(\eta, \mathcal{X}) \in \overline{J}^{2,+} u(p_0)$ with $p_0 \in \Omega$ we have

$$-\langle \mathcal{X}\eta, \eta \rangle \leq 0.$$

A *viscosity infinite harmonic* function is both a viscosity infinite subharmonic and a viscosity infinite superharmonic function.

In order to prove a comparison principle for these functions, we will employ the iterated maximum principle, which was proved in [**5**], and we will need the following lemma concerning Lipschitz functions.

Lemma 3.3. *Let u be an upper semi-continuous function and v a lower semi-continuous function in the bounded domain Ω so that at least one of u or v is Lipschitz. Let the points p and q have coordinates $p = (x_1, x_2, \ldots, x_n)$ and $q = (y_1, y_2, \ldots, y_n)$. Define the point $(p \diamond q)_i$ by*

$$(p \diamond q)_i = (x_1, x_2, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n).$$

That is, $(p \diamond q)_i$ coincides with q in the ith coordinate and coincides with p elsewhere. Then there is a finite positive constant K so that

$$\alpha_i(x_i^\alpha - y_i^\alpha)^2 \le K d_C((p \diamond q)_i, p).$$

Proof. We may assume, without loss of generality, that u is Lipschitz. By the definition of the points $p_{\vec{\alpha}}$ and $q_{\vec{\alpha}}$ in the iterated maximum principle, we have

$$u(p) - v(q) - \varphi_{\vec{\alpha}}(p, q) \le u(p_{\vec{\alpha}}) - v(q_{\vec{\alpha}}) - \varphi(p_{\vec{\alpha}}, q_{\vec{\alpha}}).$$

By setting $q = q_{\vec{\alpha}}$ and $p = (p_{\vec{\alpha}} \diamond q_{\vec{\alpha}})_i$ we arrive at

$$\varphi_{\vec{\alpha}}(p_{\vec{\alpha}}, q_{\vec{\alpha}}) - \varphi_{\vec{\alpha}}((p_{\vec{\alpha}} \diamond q_{\vec{\alpha}})_i, q_{\vec{\alpha}}) \le u(p_{\vec{\alpha}}) - u((p_{\vec{\alpha}} \diamond q_{\vec{\alpha}})_i).$$

By the definition of $\varphi_{\vec{\alpha}}$ and the Lipschitz property, we conclude there is a constant K so that

$$\frac{1}{2}\alpha_i(x_i^\alpha - y_i^\alpha)^2 \le K d_C(p_{\vec{\alpha}}, (p_{\vec{\alpha}} \diamond q_{\vec{\alpha}})_i). \qquad \square$$

We employ Jensen's auxiliary functions [**11**] and follow the outline of [**12**]. The first step is the following comparison principle.

Theorem 3.4. *Let u be an upper semi-continuous subsolution and v a lower semi-continuous supersolution to*

$$F_\varepsilon(\eta, X) = \min\{\|\eta\|^2 - \varepsilon^2, -\langle X\eta, \eta\rangle\} = 0$$

in a bounded domain Ω such that at least one of them is Lipschitz. If

$$\limsup_{q \to p} u(q) \le \liminf_{q \to p} v(q)$$

when $p \in \partial\Omega$, where both sides are not ∞ or $-\infty$ simultaneously, then

$$u(p) \le v(p)$$

for all $p \in \Omega$.

Proof. Before beginning the proof, we first note that, as in [**3**], we can construct a strict supersolution of $F_\varepsilon = 0$ called w so that $(\eta, Y) \in \overline{J}^{2,-} w(q)$ produces

$$F_\varepsilon(\eta, Y) \geq \mu(q) \geq \mu > 0.$$

We may therefore assume without loss of generality that v is a strict supersolution associated with the constant μ. We suppose that $\sup_{\overline{\Omega}}(u - v)$ occurs at the interior point p_0. Because we are using the iterated maximum principle, we need only to consider interior points by taking the α_j to be sufficiently large. Additionally, we take the α_j sufficiently large so that if $\rho_j(p_0) \neq 0$, then $\rho_j(p_{\vec{\alpha}}) \neq 0$ and $\rho_j(q_{\vec{\alpha}}) \neq 0$. We follow the procedure as in [**9**]. We have the vectors $\Upsilon_{p_{\vec{\alpha}}}$ and $\Upsilon_{q_{\vec{\alpha}}}$ defined by

$$(\Upsilon_{p_{\vec{\alpha}}})_i = \rho_i(p_{\vec{\alpha}})\alpha_i(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})$$

and

$$(\Upsilon_{q_{\vec{\alpha}}})_i = \rho_i(q_{\vec{\alpha}})\alpha_i(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}}).$$

Note that these vectors are the Grushin twist (Lemma 3.2) of the vectors formed by Euclidean differentiation. Using the construction of the vectors, we have

$$\begin{aligned} \|\Upsilon_{q_{\vec{\alpha}}}\|^2 - \|\Upsilon_{p_{\vec{\alpha}}}\|^2 &= \sum_{i=1}^{n} \alpha_i^2(\rho_i^2(q_{\vec{\alpha}}) - \rho_i^2(p_{\vec{\alpha}}))(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})^2 \\ &= \sum_{i=2}^{n} \alpha_i^2(\rho_i^2(q_{\vec{\alpha}}) - \rho_i^2(p_{\vec{\alpha}}))(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})^2 \end{aligned}$$

since $\rho_1 \equiv 1$. We now note that every term in the sum lacks an α_1. Using the fact that $\rho_i = \rho_i(x_1, x_2, \ldots, x_{i-1})$, we observe that

$$\begin{aligned} \lim_{\alpha_1 \to \infty} \alpha_2^2(\rho_2^2(q_{\vec{\alpha}}) - \rho_2^2(p_{\vec{\alpha}}))(x_2^{\vec{\alpha}} - y_2^{\vec{\alpha}})^2 &= 0 \\ \lim_{\alpha_2 \to \infty} \lim_{\alpha_1 \to \infty} \alpha_3^2(\rho_3^2(q_{\vec{\alpha}}) - \rho_3^2(p_{\vec{\alpha}}))(x_3^{\vec{\alpha}} - y_3^{\vec{\alpha}})^2 &= 0 \\ \vdots \qquad\qquad & \vdots \\ \lim_{\alpha_{n-1} \to \infty} \lim_{\alpha_{n-2} \to \infty} \cdots \lim_{\alpha_1 \to \infty} \alpha_n^2(\rho_n^2(q_{\vec{\alpha}}) - \rho_n^2(p_{\vec{\alpha}}))(x_n^{\vec{\alpha}} - y_n^{\vec{\alpha}})^2 &= 0. \end{aligned}$$

We then are able to conclude that

$$\lim_{\alpha_n\to\infty}\lim_{\alpha_{n-1}\to\infty}\cdots\lim_{\alpha_2\to\infty}\lim_{\alpha_1\to\infty}\|\Upsilon_{q_{\vec{\alpha}}}\|^2-\|\Upsilon_{p_{\vec{\alpha}}}\|^2=0. \tag{3.1}$$

Turning to the matrices $X^{\vec{\alpha}}$ and $Y^{\vec{\alpha}}$ in the second order Euclidean jets, we use Lemma 3.2 to construct the matrices $\mathcal{X}^{\vec{\alpha}}$ and $\mathcal{Y}^{\vec{\alpha}}$ by

$$\mathcal{X}_{ij}^{\vec{\alpha}}=\begin{cases}\rho_i(p_{\vec{\alpha}})\rho_j(p_{\vec{\alpha}})X_{ij}^{\vec{\alpha}}+(1/2)(\partial\rho_j/\partial x_i)(p_{\vec{\alpha}})\rho_i(p_{\vec{\alpha}})\alpha_j(x_j^{\vec{\alpha}}-y_j^{\vec{\alpha}}) & i\leq j,\\ \mathcal{X}_{ji}^{\vec{\alpha}} & i>j\end{cases}$$

and

$$\mathcal{Y}_{ij}^{\vec{\alpha}}=\begin{cases}\rho_i(q_{\vec{\alpha}})\rho_j(q_{\vec{\alpha}})Y_{ij}^{\vec{\alpha}}+(1/2)(\partial\rho_j/\partial x_i)(q_{\vec{\alpha}})\rho_i(q_{\vec{\alpha}})\alpha_j(x_j^{\vec{\alpha}}-y_j^{\vec{\alpha}}) & i\leq j,\\ \mathcal{Y}_{ji}^{\vec{\alpha}} & i>j.\end{cases}$$

We then have

$$(\Upsilon_{p_{\vec{\alpha}}},\mathcal{X}^{\vec{\alpha}})\in\overline{J}^{2,+}u(p_{\vec{\alpha}})$$

and

$$(\Upsilon_{q_{\vec{\alpha}}},\mathcal{Y}^{\vec{\alpha}})\in\overline{J}^{2,+}u(q_{\vec{\alpha}}).$$

Given vectors ϵ and χ, the matrices $X^{\vec{\alpha}}$ and $Y^{\vec{\alpha}}$ satisfy the relation [**9**]

$$\langle X^{\vec{\alpha}}\epsilon,\epsilon\rangle_{\text{eucl}}-\langle Y^{\vec{\alpha}}\kappa,\kappa\rangle_{\text{eucl}}\leq\langle C\chi,\chi\rangle_{\text{eucl}}$$

where the vector $\chi=(\epsilon,\kappa)$ and the matrix C is a block matrix of the form

$$\begin{pmatrix}B & -B\\ -B & B\end{pmatrix}$$

with the submatrix B defined by

$$B_{ij}=\begin{cases}\alpha_i+\sigma 2\alpha_i^2 & i=j\\ 0 & i\neq j\end{cases}$$

for a fixed small constant σ.

Using the construction of the matrices, we are able to compute

$$\begin{aligned}
&\langle \mathcal{X}^{\vec{\alpha}}\Upsilon_{p_{\vec{\alpha}}}, \Upsilon_{p_{\vec{\alpha}}}\rangle - \langle \mathcal{Y}^{\vec{\alpha}}\Upsilon_{q_{\vec{\alpha}}}, \Upsilon_{q_{\vec{\alpha}}}\rangle \\
&\quad \leq \langle B(\widetilde{\Upsilon_{p_{\vec{\alpha}}}} - \widetilde{\Upsilon_{q_{\vec{\alpha}}}})\widetilde{\Upsilon_{p_{\vec{\alpha}}}} - \widetilde{\Upsilon_{q_{\vec{\alpha}}}}\rangle + \sum_{j=1}^{n}\sum_{i<j}\alpha_j(x_j^{\vec{\alpha}} - y_j^{\vec{\alpha}}) \\
&\qquad \times \left(\left(\frac{\partial \rho_j}{\partial x_i}\rho_i\right)(p_{\vec{\alpha}})(\Upsilon_{p_{\vec{\alpha}}})_i(\Upsilon_{p_{\vec{\alpha}}})_j - \left(\frac{\partial \rho_j}{\partial x_i}\rho_i\right)(q_{\vec{\alpha}})(\Upsilon_{q_{\vec{\alpha}}})_i(\Upsilon_{q_{\vec{\alpha}}})_j\right)
\end{aligned}$$

where $\widetilde{\Upsilon_{p_{\vec{\alpha}}}}$ is the Grushin twist (Lemma 3.2) of the vector $\Upsilon_{p_{\vec{\alpha}}}$, see also [**5**]. Therefore, the matrix difference can be expressed as

$$\begin{aligned}
&\sum_{i=1}^{n}(\alpha_i + \sigma 2\alpha_i^2)(\rho_i^2(p_{\vec{\alpha}}) - \rho_i^2(q_{\vec{\alpha}}))^2\alpha_i^2(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})^2 \\
&\qquad\qquad + \sum_{j=1}^{n}\sum_{i<j}\alpha_j^2\alpha_i(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})(x_j^{\vec{\alpha}} - y_j^{\vec{\alpha}})^2 \\
&\qquad\qquad \times \left(\left(\frac{\partial \rho_j}{\partial x_i}\rho_i^2\rho_j\right)(p_{\vec{\alpha}}) - \left(\frac{\partial \rho_j}{\partial x_i}\rho_i^2\rho_j\right)(q_{\vec{\alpha}})\right).
\end{aligned}$$

Using the fact that $\rho_1 \equiv 1$, we see that the term corresponding to $i = 1$ in the first sum and the terms corresponding to $j = 1$ in the second sum are zero. Proceeding as in the vector difference estimate, we observe that the first sum has no α_1 terms and the construction of the polynomials again produces

$$\begin{aligned}
\lim_{\alpha_1\to\infty}(\alpha_2 + \sigma 2\alpha_2^2)(\rho_2^2(p_{\vec{\alpha}}) - \rho_2^2(q_{\vec{\alpha}}))^2\alpha_2^2(x_2^{\vec{\alpha}} - y_2^{\vec{\alpha}})^2 &= 0 \\
\lim_{\alpha_2\to\infty}\lim_{\alpha_1\to\infty}(\alpha_3 + \sigma 2\alpha_3^2)(\rho_3^2(p_{\vec{\alpha}}) - \rho_3^2(q_{\vec{\alpha}}))^2\alpha_3^2(x_3^{\vec{\alpha}} - y_3^{\vec{\alpha}})^2 &= 0 \\
&\vdots \\
\lim_{\alpha_{n-1}\to\infty}\cdots\lim_{\alpha_1\to\infty}(\alpha_n + \sigma 2\alpha_n^2)(\rho_n^2(p_{\vec{\alpha}}) - \rho_n^2(q_{\vec{\alpha}}))^2\alpha_n^2(x_n^{\vec{\alpha}} - y_n^{\vec{\alpha}})^2 &= 0
\end{aligned}$$

so that we may conclude that

$$\lim_{\alpha_n\to\infty}\cdots\lim_{\alpha_1\to\infty}\sum_{i=1}^{n}(\alpha_i + \sigma 2\alpha_i^2)(\rho_i^2(p_{\vec{\alpha}}) - \rho_i^2(q_{\vec{\alpha}}))^2\alpha_i^2(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})^2 = 0.$$

We now turn to the second sum. First, let us consider the term where $j=2$ (which forces $i=1$). We note that

$$\left(\frac{\partial\rho_2}{\partial x_1}\rho_1^2\rho_2\right)(p_{\vec{\alpha}})-\left(\frac{\partial\rho_2}{\partial x_1}\rho_1^2\rho_2\right)(q_{\vec{\alpha}})\sim(x_1^{\vec{\alpha}}-y_1^{\vec{\alpha}}),$$

and so we obtain

$$\lim_{\alpha_1\to\infty}\alpha_2^2\alpha_1(x_1^{\vec{\alpha}}-y_1^{\vec{\alpha}})(x_2^{\vec{\alpha}}-y_2^{\vec{\alpha}})^2\left(\left(\frac{\partial\rho_2}{\partial x_1}\rho_2\right)(p_{\vec{\alpha}})-\left(\frac{\partial\rho_2}{\partial x_1}\rho_2\right)(q_{\vec{\alpha}})\right)=0.$$

Next, we consider the terms where $j>2$. We denote

$$T_{ij}=\alpha_j^2\alpha_i(x_i^{\vec{\alpha}}-y_i^{\vec{\alpha}})(x_j^{\vec{\alpha}}-y_j^{\vec{\alpha}})^2\left(\left(\frac{\partial\rho_j}{\partial x_i}\rho_i^2\rho_j\right)(p_{\vec{\alpha}})-\left(\frac{\partial\rho_j}{\partial x_i}\rho_i^2\rho_j\right)(q_{\vec{\alpha}})\right).$$

Since $i<j$, we can easily control

$$S_{ij}\stackrel{\text{def}}{\equiv}\lim_{\alpha_{i-1}\to\infty}\lim_{\alpha_{i-2}\to\infty}\cdots\lim_{\alpha_1\to\infty}T_{ij}$$

through the polynomials since T_{ij} contains only α_i and α_j. In particular, by the triangular definition of the polynomials, we have

$$S_{ij}=\alpha_j^2\alpha_i(x_i^{\vec{\alpha}}-y_i^{\vec{\alpha}})(x_j^{\vec{\alpha}}-y_j^{\vec{\alpha}})^2(\rho_i(p_0))^2\left(\left(\frac{\partial\rho_j}{\partial x_i}\rho_j\right)(p_{\vec{\alpha}})-\left(\frac{\partial\rho_j}{\partial x_i}\rho_j\right)(q_{\vec{\alpha}})\right).$$

Thus, if $\rho_i(p_0)=0$, then $S_{ij}=0$ and

$$\lim_{\alpha_n\to\infty}\lim_{\alpha_{n-1}\to\infty}\cdots\lim_{\alpha_i\to\infty}S_{ij}=0.$$

If $\rho_i(p_0)\neq 0$, then by our initial assumption, $\rho_i(p_{\vec{\alpha}})\neq 0$ and $\rho_i(q_{\vec{\alpha}})\neq 0$. In particular, $x_i^{\vec{\alpha}}$ and $y_i^{\vec{\alpha}}$ lie in a locally Riemannian neighborhood of x_i^0. Lemma 3.3 and equation (2.2) then can be used to produce a constant K so that

$$\alpha_i(x_i^{\vec{\alpha}}-y_i^{\vec{\alpha}})^2\leq Kd_C(p_{\vec{\alpha}},(p_{\vec{\alpha}}\diamond q_{\vec{\alpha}})_i)\sim|x_i^{\vec{\alpha}}-y_i^{\vec{\alpha}}|.$$

Using this estimate, we have

$$\lim_{\alpha_i\to\infty}S_{ij}\sim\alpha_j^2(x_j^{\vec{\alpha}}-y_j^{\vec{\alpha}})^2(\rho_i(p_0))^2\left(\left(\frac{\partial\rho_j}{\partial x_i}\rho_i^2\rho_j\right)(p_{\vec{\alpha}})-\left(\frac{\partial\rho_j}{\partial x_i}\rho_i^2\rho_j\right)(q_{\vec{\alpha}})\right).$$

Again, the definition of the polynomials ρ_j results in

$$\lim_{\alpha_{j-1}\to\infty}\lim_{\alpha_{j-2}\to\infty}\cdots\lim_{\alpha_i\to\infty} S_{ij} = 0.$$

Combining these results, we have

$$\lim_{\alpha_n\to\infty}\lim_{\alpha_{n-1}\to\infty}\cdots\lim_{\alpha_2\to\infty}\lim_{\alpha_1\to\infty} T_{ij} = 0,$$

and so we conclude

$$\lim_{\alpha_n\to\infty}\lim_{\alpha_{n-1}\to\infty}\cdots\lim_{\alpha_2\to\infty}\lim_{\alpha_1\to\infty}\langle \mathcal{X}^{\vec{\alpha}}\Upsilon_{p_{\vec{\alpha}}},\Upsilon_{p_{\vec{\alpha}}}\rangle - \langle \mathcal{Y}^{\vec{\alpha}}\Upsilon_{q_{\vec{\alpha}}},\Upsilon_{q_{\vec{\alpha}}}\rangle = 0.$$

Because u is a viscosity subsolution at $p_{\vec{\alpha}}$ and v is a strict viscosity supersolution at $q_{\vec{\alpha}}$, we have

$$F_\varepsilon(\Upsilon_{p_{\vec{\alpha}}},\mathcal{X}^{\vec{\alpha}}) \leq 0$$

and

$$F_\varepsilon(\Upsilon_{q_{\vec{\alpha}}},\mathcal{Y}^{\vec{\alpha}}) \geq \mu > 0.$$

We then subtract the two equations to obtain

$$\begin{aligned} 0 < \mu &\leq F_\varepsilon(\Upsilon_{q_{\vec{\alpha}}},\mathcal{Y}^{\vec{\alpha}}) - F_\varepsilon(\Upsilon_{p_{\vec{\alpha}}},\mathcal{X}^{\vec{\alpha}}) \\ &= \max\{\|\Upsilon_{q_{\vec{\alpha}}}\|^2 - \|\Upsilon_{p_{\vec{\alpha}}}\|^2, \langle \mathcal{X}^{\vec{\alpha}}\Upsilon_{p_{\vec{\alpha}}},\Upsilon_{p_{\vec{\alpha}}}\rangle - \langle \mathcal{Y}^{\vec{\alpha}}\Upsilon_{q_{\vec{\alpha}}},\Upsilon_{q_{\vec{\alpha}}}\rangle\}. \end{aligned}$$

We then arrive at a contradiction via equations (3.1) and (3.2) and applying the iterated limit. □

Uniqueness of infinite harmonic functions then follows as in [**12**], producing the comparison principle for viscosity infinite harmonic functions. Namely,

Theorem 3.5. *Let u be a viscosity infinite subharmonic function, and let v be a viscosity infinite superharmonic function in a domain Ω such that if $p \in \partial\Omega$,*

$$\limsup_{q\to p} u(q) \leq \limsup_{q\to p} v(q)$$

where both sides are not $-\infty$ *or* $+\infty$ *simultaneously. Then* $u \leq v$ *in* Ω*.*

4. Properties of viscosity infinite harmonic functions. Having proved the comparison principle, we now present two consequences that will enable us to prove the Main theorem.

Definition 3. A lower semi-continuous function $u : \Omega \to \mathbf{R} \cup \{\infty\}$ that is not identically infinity in each component of Ω is *potential superharmonic* if for each open set $U \subset\subset \Omega$ and each viscosity infinite harmonic function f defined on U,

$$u \geq f \quad \text{on } \partial U \Longrightarrow u \geq f \quad \text{in } U.$$

A function u is *potential subharmonic* if $-u$ is potential superharmonic. A function u is *potential harmonic* if it is both potential subharmonic and superharmonic.

We then have the first consequence of the comparison principle, namely,

Lemma 4.1. *A function u is viscosity infinite subharmonic if and only if it is potential subharmonic. A function u is viscosity infinite superharmonic if and only if it is potential superharmonic. A function is viscosity infinite harmonic if and only if it is potentially harmonic.*

Proof. The last statement follows from the first two. We first prove the second statement. Suppose u is viscosity infinite superharmonic in Ω. By the comparison principle, if $f \leq u$ on ∂U, $U \subset\subset \Omega$, then $f \leq u$ in U. Thus, u is potential superharmonic.

Suppose u is not viscosity infinite superharmonic. Then, there is a point $p_0 \in \Omega$ and a C^2_{sub} function ϕ so that $u(p_0) = \phi(p_0)$ and $u(p) > \phi(p)$ in a neighborhood of p_0 with

$$-\Delta_\infty \phi(p_0) < 0.$$

By continuity, this inequality holds in a ball $B(p_0, r)$ for sufficiently small r. Let

$$m = \inf_{p \in \partial B} (u(p) - \phi(p)).$$

By lower semi-continuity of the functions and compactness of ∂B, we have $m > 0$. Define the function

$$\psi \equiv \phi + \frac{m}{2},$$

and let v be the unique viscosity infinite harmonic function in B with boundary data ψ. Since ψ is a subsolution in B, we have $\psi \leq v$ in B. In addition, we have $u(p_0) < \psi(p_0)$ so that

$$u \ngeq v \quad \text{in } B.$$

However, by construction, $u \geq v$ on ∂B, so that u is not potential superharmonic. The proof of the first statement is similar and omitted. □

Next, we recall the definition of absolute minimizers.

Definition 4. The function $u \in \mathrm{Lip}\,(\overline{\Omega})$ is an *absolute minimizer* if for every $V \subset \Omega$ and $v \in \mathrm{Lip}\,(V)$, such that $u = v$ on ∂V, then

$$\|\nabla_0 u\|_{L^\infty(V)} \leq \|\nabla_0 v\|_{L^\infty(V)}.$$

The function $u \in \mathrm{Lip}\,(\overline{\Omega})$ is an *absolute superminimizer* if the above holds for $v \geq u$. The function $u \in \mathrm{Lip}\,(\overline{\Omega})$ is an *absolute subminimizer* if the above holds for $v \leq u$.

It is clear from the definition that an absolute minimizer is both an absolute subminimizer and absolute superminimizer. The proof of the converse is given in [**1**, Section 4.3]. Also, a function u is a subminimizer exactly when $-u$ is a super-minimizer. It was shown in [**6**] (for C^1_{sub}) and [**15**] (for arbitrary) that absolute minimizers in Grushin-type spaces are viscosity infinite harmonic functions. In particular, the proof shows that absolute subminimizers are viscosity infinite subharmonic. We then have the following lemma, whose proof is similar to that in [**13**] and omitted.

Lemma 4.2. *Given a domain Ω, let u be an infinite harmonic function in Ω. Then $u \in W^{1,\infty}_{\text{loc}}$ and it is an absolute minimizer with respect to its trace.*

The results of this section can be summarized by the following theorem, which gives us part of the Main theorem.

Theorem 4.3. *Given a domain Ω and a function u, the following are equivalent.*

(1) *u is an absolute minimizer.*

(2) *u is viscosity infinite harmonic.*

(3) *u is potential harmonic.*

We also have what [**1**] refers to as "one-sided results." Namely,

Theorem 4.4. *Given a domain Ω and an absolute sub (super)-minimizer u, then u is viscosity sub (super)-harmonic. In addition, u is viscosity infinite sub (super)-harmonic if and only if it is potential sub (super)-harmonic.*

5. Grushin cones. In this section, we discuss Grushin cones and extend results found in [**1, 13**] including the important property of comparison with cones. This will complete the proof of the Main theorem. In the Euclidean case, functions enjoying comparison with cones were shown to be exactly those that were viscosity infinite harmonic [**10**], and we extend this result to Grushin-type spaces. We begin with the definition of Grushin cones.

Definition 5. Let $a, b \in \mathbf{R}$. Given a point p and an open set U, we define the function $d : \partial(U \setminus \{p\}) \to \mathbf{R}$ by

$$d(q) = a + b\, d_C(p, q).$$

The Grushin cone based on (U, p) is the unique viscosity infinite harmonic function $\omega^{a,b}_{U,p}$ in $U \setminus \{p\}$ such that

$$\omega^{a,b}_{U,p} = d \quad \text{on } \partial(U \setminus \{p\}).$$

To obtain an upper bound on cones, we state a result of Monti and Serra-Cassano [**14**].

Theorem 5.1. *Given $q \in G_n$, for almost every $p \in G_n$, we have*

$$\|\nabla_0 d_C(p,q)\| \le 1.$$

Using this theorem, we are able to find bounds for the pointwise values of Grushin cones via the following proposition.

Proposition 5.2. *Given a pair (U,p) and $a, b \in \mathbf{R}$, the cone $\omega_{U,p}^{a,b}$ satisfies*

$$\omega_{U,p}^{a,b}(q) \le a + \operatorname{abs}(b)\, d_C(p,q)$$
$$\omega_{U,p}^{a,b}(q) \ge a - \operatorname{abs}(b)\, d_C(p,q)$$

for $q \in U$. Here, $\operatorname{abs}(\cdot)$ *denotes absolute value.*

Proof. If $p \in \overline{U}$, we compute

$$\omega_{U,p}^{a,b}(q) - a = \omega_{U,p}^{a,b}(q) - \omega_{U,p}^{a,b}(p) \le \|\nabla_0 \omega_{U,p}^{a,b}\|_{L^\infty(U)} d_C(p,q).$$

However, the cones, as viscosity infinite harmonic functions, are absolute minimizers. Thus,

$$\|\nabla_0 \omega_{U,p}^{a,b}\|_{L^\infty(U)} \le \|b \nabla_0 d_C(p,q)\|_{L^\infty(U)} \le \operatorname{abs}(b).$$

We also note that we have

$$\omega_{U,p}^{a,b}(q) - a = \omega_{U,p}^{a,b}(q) - \omega_{U,p}^{a,b}(p) \ge -\|\nabla_0 \omega_{U,p}^{a,b}\|_{L^\infty(U)} d_C(p,q).$$

Similarly, we obtain

$$\omega_{U,p}^{a,b}(q) - a \ge -\operatorname{abs}(b) d_C(p,q).$$

If $p \notin \overline{U}$, let γ be the geodesic between p and q, and observe that there is a point $q' \in \partial U \cap \gamma$. Thus,

$$d_C(p,q) = d_C(p,q') + d_C(q',q).$$

In particular, we have

$$\omega_{U,p}^{a,b}(q)-a-b\ d_C(p,q')=\omega_{U,p}^{a,b}(q)-\omega_{U,p}^{a,b}(q')\leq\|\nabla_0\omega_{U,p}^{a,b}\|_{L^\infty(U)}d_C(q,q').$$

As above, we then obtain

$$\begin{aligned}\omega_{U,p}^{a,b}(q)&\leq a+b\ d_C(p,q')+\operatorname{abs}(b)d_C(q,q')\\&\leq a+\operatorname{abs}(b)(d_C(p,q')+d_C(q,q')).\end{aligned}$$

We also then have

$$\omega_{U,p}^{a,b}(q)-a-b\ d_C(p,q')=\omega_{U,p}^{a,b}(q)-\omega_{U,p}^{a,b}(q')\geq-\|\nabla_0\omega_{U,p}^{a,b}\|_{L^\infty(U)}d_C(q,q').$$

We then conclude

$$\omega_{U,p}^{a,b}(q)\geq a-\operatorname{abs}(b)d_C(p,q')-\operatorname{abs}(b)d_C(q,q').$$

The results then follow from our choice of q'. □

We are now ready to define the concept of comparison with cones, analogous to the Euclidean case [**10**].

Definition 6. Let $U\subset\mathbf{R}^n$ be an open set, and let $u:U\to\mathbf{R}$. Then u enjoys comparison with cones from above in U if for every open $V\subset U$ and $a,b\in\mathbf{R}$ for which

$$u\leq\omega_{U,p}^{a,b}$$

holds on $\partial(V\setminus\{p\})$, then we have

$$u\leq\omega_{U,p}^{a,b}$$

in V. A similar definition holds for the function u enjoying comparison with cones from below in U. The function u enjoys comparison with cones in U exactly when it enjoys comparison with cones from above and below.

With these definitions, we obtain the following implication.

Lemma 5.3. *A viscosity infinite superharmonic function in U enjoys comparison with cones from below in U. Similarly, a viscosity infinite subharmonic function enjoys comparison with cones from above in U and a viscosity infinite harmonic function enjoys comparison with cones in U.*

Proof. The first and second statements are symmetric and the last statement follows from the first two. We therefore will consider only the first statement. Suppose first that u is viscosity infinite superharmonic. Let $\omega_{U,p}^{a,b}$ be a cone so that

$$\omega_{U,p}^{a,b} \leq u \quad \text{on } \partial(V \setminus \{p\})$$

for some $V \subset U$. Then, by the comparison principle, Theorem 3.5, we have

$$\omega_{U,p}^{a,b} \leq u \quad \text{in } V.$$

Thus, u enjoys comparison with cones from below. □

Next, we focus our attention on functions that enjoy comparison with cones from above. We will employ results from [**1**]. However, some of the proofs must be altered to accommodate for the differences between Euclidean and Grushin geometry and the lack of an explicit formula for the Grushin cones. Those differences are highlighted where they appear.

Proposition 5.4. *Let u be an upper semi-continuous function in a domain U that enjoys comparison with cones from above. For a point $q \in U$, define the function $S(q,r)$ by*

$$S(q,r) \equiv \max\left\{\frac{u(w)-u(q)}{r} : d_C(w,q) = r\right\}.$$

Then, we have

(1) $\max\{u(p) : d_C(p,q) = r\} = \max\{u(p) : d_C(p,q) \leq r\}$.

(2) $u(p) \leq u(q) + S(q,r)d_C(p,q)$ *for points $p \in U$ so that $d_C(p,q) \leq r$ with $0 \leq r < d_C(q, \partial U)$,*

(3) *$S(q,r)$ is monotonic and nonnegative,*

(4) $u \in W^{1,\infty}_{\text{loc}}(U)$.

Proof. To prove equation (1), we note that the cone with boundary data $M \equiv \max\{u(p) : d_C(p,q) = r\}$ on $B(q,r)$ is the constant M itself. Since $u \leq M$ on $\partial B(q,r)$, comparison with cones from above implies

$$u \leq M \quad \text{in } B(q,r).$$

Equation (1) follows. To prove equation (2), we note that it holds when $d_C(p,q) = 0$ and when $d_C(p,q) = r$. Using $u(q) + S(q,r)d_C(p,q)$ as the boundary data for the cone ω, we have by comparison with cones from above that

$$u(p) \leq \omega(p).$$

Equation (2) then follows from Proposition 5.2. The proof of statement (3) is identical to Lemma 2.4 of [**10**] and the proof of statement (4) is identical to Lemma 2.5 of [**10**] and therefore omitted. □

In light of the previous proposition, it is reasonable to define the function $S(q)$ by

$$S(q) \equiv \lim_{r \to 0} S(q,r) = \inf\{S(q,r) : 0 < r < d_C(q, \partial U)\}.$$

In addition, if we let $L_u(U)$ denote the smallest constant L so that

$$|u(p) - u(q)| \leq L\, d_C(p,q)$$

for $p, q \in U$, then we may define the similar function $T_u(p)$ by

$$T_u(p) \equiv \lim_{r \to 0} L_u(B(p,r)) = \inf\{L_u(B(p,r)) : 0 < r < d_C(p, \partial U)\}.$$

We note that this is well defined on the extended reals and if u is Lipschitz, we have

$$\sup_{p \in U} T_u(p) = \|\nabla_0 u\|_{L^\infty}.$$

We then have the following proposition concerning properties of $S(q)$ and $T_u(p)$.

Proposition 5.5. *Let u be as in the previous proposition. Then, we have*

(1) $T_u(p)$ *is upper semi-continuous.*

(2) *Let $p, q \in U$ so that $\gamma_{pq} \subset U$ where γ_{pq} is the geodesic between p and q with $\gamma_{pq}(0) = p$ and $\gamma_{pq}(1) = q$. Then,*

$$|u(p) - u(q)| \leq (\max\{S(w) : w \in \gamma_{pq}\})d_C(p, q).$$

(3) $T_u(q) = S(q)$.

Proof. The proof of the first statement can be found in [**1**, Section 1.5]. The next two statements follow from Lemma 2.15 in [**1**] and the fact that in the Grushin environment, we still have

$$d_C(p, \gamma_{pq}(t)) = td_C(p, q)$$

along the geodesic. □

Lastly, we state a technical lemma, which is Proposition 4.7 of [**1**]. The proof is omitted.

Lemma 5.6. *Let U be bounded, and let $u \in C(\overline{U})$ enjoy comparison with cones from above. Let $p_0 \in U$ so that $S(p_0) > 0$, and let $\delta > 0$. Then, there is a sequence of points $\{p_j\} \subset U$ and a point $p_\infty \in \partial U$ so that*

(1) $d_C(p_j, p_{j-1}) \leq \delta$,

(2) γ_j, *the geodesic between p_j and p_{j-1} is contained in U,*

(3) $S(p_j) \geq S(p_{j-1})$,

(4) $\lim_{j\to\infty} p_j = p_\infty$,

(5) $u(p_\infty) - u(p_0) \geq S(p_0) \sum_{j=1}^{\infty} d_C(p_j, p_{j-1})$.

We now are able to prove the following theorem.

Theorem 5.7. *A function that enjoys comparison with cones from above is an absolute subminimizer. A function that enjoys comparison* *with cones from below is an absolute superminimizer. A function that enjoys comparison with cones is an absolute minimizer.*

Proof. Again, the last statement follows from the first two, and the proof of the first two are symmetric, so we shall prove only the first statement. We suppose that u enjoys comparison with cones from above and is not an absolute subminimizer in the domain U. Let v be a Lipschitz function so that $v \leq u$ in U and $u = v$ on ∂U. Suppose that

$$\|\nabla_0 u\|_{L^\infty(V)} > \|\nabla_0 v\|_{L^\infty(V)}.$$

This is equivalent to

$$\sup\{T_u(p) : p \in U\} > \sup\{T_v(p) : p \in U\}.$$

Thus, there is a $p_0 \in U$ so that

$$T_u(p_0) > \sup\{T_v(p) : p \in U\}.$$

using the lemmas above, we then have

$$\begin{aligned}
u(p_\infty) - u(p_0) &\geq S(p_0) \sum_{j=1}^{\infty} |p_j - p_{j-1}| \\
&= T_u(p_0) \sum_{j=1}^{\infty} |p_j - p_{j-1}| \\
&> \sup\{T_v(p) : p \in U\} \sum_{j=1}^{\infty} |p_j - p_{j-1}| \\
&\geq \sum_{j=1}^{\infty} |v(p_j) - v(p_{j-1})| \\
&\geq \sum_{j=1}^{\infty} (v(p_j) - v(p_{j-1})) \\
&= v(p_\infty) - v(p_0).
\end{aligned}$$

Since $p_\infty \in \partial U$ and $v = u$ on ∂U, we then obtain

$$u(p_0) < v(p_0)$$

contradicting the fact that $v \leq u$ in U. Our supposition is therefore false, and u is an absolute subminimizer. □

Combining Theorem 4.3, Theorem 4.4, Lemma 5.3 and Theorem 5.7, we obtain the following theorem.

Main theorem. *Given a domain Ω and a function u, the following are equivalent.*

(1) *u is an absolute minimizer.*

(2) *u is viscosity infinite harmonic.*

(3) *u is potential harmonic.*

(4) *u enjoys comparison with cones.*

In addition, the corresponding "one-sided" statements hold. Namely, the following are equivalent.

(I) *u is an absolute sub (super)-minimizer.*

(II) *u is viscosity infinite sub (super)-harmonic.*

(III) *u is potential infinite sub (super)-harmonic.*

(IV) *u enjoys comparison with Grushin cones from above (below).*

This theorem then provides us with a tool to study the Grushin distance function. Precisely, we have the following corollary.

Corollary 5.8. *Let $a, b \in \mathbf{R}$ with $b \geq 0$. Let U be a domain and p an arbitrary point. Define the function $d : U \to \mathbf{R}$ by $d(q) = a + b\, d_C(p, q)$. Then $d(q)$ is a viscosity infinite superharmonic function. In particular, the distance function is a viscosity infinite superharmonic function. By symmetry, $d_-(q) \equiv a - b\, d_C(p, q)$ is a viscosity infinite subharmonic function.*

Proof. We will show the function $d(q)$ enjoys comparison with cones from below. Let U, p, a, b and $d(q)$ be as above. Let ω_d be the Grushin cone equal to $d(q)$ on $\partial(U \setminus \{p\})$. Suppose the cone ω has the property that $\omega \leq d(q) = \omega_d$ on ∂U. Then, by the comparison principle and

Proposition 5.2,

$$\omega \leq \omega_d \leq a + b\, d_C(p,q) = d(q)$$

in U. □

6. Harnack inequality. Before examining the geometry of Grushin cones, we will need the Harnack inequality and its consequences for viscosity infinite harmonic functions in Grushin spaces. The proof of the following theorem is standard and omitted.

Theorem 6.1. *Suppose that u is a nonnegative viscosity infinite harmonic function in a domain Ω. Then for all $\zeta \in C_0^\infty(\Omega)$, we have*

$$\|\zeta \nabla_0 \log u\|_{L^\infty(\Omega)} \leq \|\nabla_0 \zeta\|_{L^\infty(\Omega)}.$$

Let $B(r)$ and $B(R)$ be concentric balls of radius r and R, respectively. It is easy to construct a function ζ so that $\zeta = 1$ on $B(r)$, $\zeta = 0$ outside $B(R)$ and $\|\nabla_0 \zeta\| \leq C/(R-r)$. From Theorem 6.1 we obtain

$$\|\nabla_0 \log u\|_{L^\infty(B(r))} \leq CR - r.$$

This implies the following corollary.

Corollary 6.2 (Harnack inequality). *Let u be nonnegative viscosity infinite harmonic in a domain Ω. Let B be a ball so that $2B \subset \Omega$. Then, there is a constant C so that*

$$\sup_B u \leq C \inf_B u.$$

In particular, we have

$$u(p) \leq \exp\left\{ C \left(\frac{d_C(p,q)}{R-r} \right) \right\} u(q).$$

Finally, we state immediate consequences of the Harnack inequality.

Corollary 6.3. *The only infinite harmonic functions bounded from below in the whole Grushin-type space are the constants.*

Corollary 6.4 (Strong maximum principle). *A nonconstant infinite harmonic function in a domain Ω cannot attain its supremum or infimum.*

7. Geometry of Grushin cones. In the Euclidean environment, it is well known ([**1, 10**]) that the functions $d(q)$ are also viscosity infinite subharmonic functions and thus the Euclidean cones are exactly those functions $d(q)$. By symmetry, the restriction that $b \geq 0$ can be removed, and the result holds for all $d(q)$. Due to the richness of the geometry of Grushin-type spaces, the analogous result in Grushin-type spaces does not necessarily hold. (For a deeper discussion on the geometry of certain Grushin-type spaces, see [**7, 8**] and the references therein.)

We begin with two geometric definitions concerning points in a domain U.

Definition 7. Let U be a bounded domain, and let p be an arbitrary point.

(1) A point $q \in U$ is *geodesically near with respect to the point p* if

$$q \in \Lambda = \left\{ \bigcup_{z \in \partial(U \setminus \{p\})} \gamma : \gamma \quad \text{is a geodesic between } p \text{ and } z \right\}.$$

(2) A point $q \in U$ that is not geodesically near is *geodesically far with respect to the point p*. That is, $y \notin \Lambda$.

(3) A point $q \in U$ is *boundary near with respect to the point p* if there exists a $z \in \partial U$ so that

$$d_C(p, q) < d_C(p, z).$$

(4) A point $q \in U$ that is not boundary near is *boundary far with respect to the point p*. That is, for all $z \in \partial U$, we have

$$d_C(p, q) \geq d_C(p, z).$$

We drop the phrase "with respect to p" in these definitions when the point p is understood.

We first note that, because geodesics need not be unique ([**7, 8**]), the set Λ actually includes *all* geodesics between points p and z. Points that are geodesically near with respect to p lie on *some* geodesic from p to the boundary point z. Additionally, it is clear that geodesically near implies boundary near, or equivalently, boundary far implies geodesically far. We next note that, unlike the Euclidean case, interior points need not be geodesically near.

We first consider cones with constant boundary data. In the case when $b = 0$, we have $\omega_d(q) = d(q) = a$ for all points q in any bounded domain U. In the case when $b > 0$, the constant boundary data and uniqueness of the cones produces the constant cone ω_d. We have the following theorem concerning constant Grushin cones when $b > 0$.

Theorem 7.1. *Let U be a bounded domain and $a, b \in \mathbf{R}$ with $b > 0$. Define $d(q) = a + b\ d_C(p, q)$ as above. Suppose $d(z) = K$ for $z \in \partial(U \setminus \{p\})$ for some constant K. Let ω_d be the (constant) Grushin cone with boundary data K. Then the point $q \in U$ is boundary far with respect to p exactly when $\omega_d(q) < d(q)$.*

Proof. Suppose that q is a boundary far with respect to p. Because q is an interior point to $U \setminus \{p\}$, there is an $r > 0$ so that the ball $B(q, r) \subset (U \setminus \{p\})$. Let γ be a geodesic from p to q. Then, there is a point $\hat{p} \in (B(q, r) \setminus \{q\}) \cap \gamma$ with the property

$$d_C(p, q) = d_C(p, \hat{p}) + d_C(\hat{p}, q).$$

Using this property, we see that $d(q) > d(\hat{p})$. Suppose that $\omega_d(q) = d(q)$. We would then have

$$d(q) = \omega_d(q) = K = \omega_d(\hat{p}) \leq d(\hat{p}) < d(q).$$

We note that the penultimate inequality is a consequence of Proposition 5.2 and therefore conclude that $\omega_d(q) < d(q)$.

Suppose next that $\omega_d(q) < d(q)$. Then by Proposition 5.2, we have

$$K = \omega_d(q) < d(q).$$

That is, for any $z \in \partial(U \setminus \{p\})$,

$$a + b\, d_C(p, z) < a + b\, d_C(p, q).$$

Because $b > 0$, we conclude that q is a boundary far with respect to p. □

The case of nonconstant cones is more involved. We have the following partial result that parallels the constant case.

Theorem 7.2. *Let U, p, a, b be as in Theorem 7.1. Suppose that $d(z)$ is nonconstant on $\partial(U \setminus \{p\})$, and let ω_d be the (nonconstant) Grushin cone with boundary data $d(z)$. Then we have the implications*

$$q \quad \text{is boundary far with respect to } p \Longrightarrow \\ \omega_d(q) < d(q) \Longrightarrow q \quad \text{is geodesically far with respect to } p.$$

Proof. We first observe that as a nonconstant (continuous) infinite harmonic function on a compact set, we have that ω_d achieves its maximum on $\overline{U}$, and by Corollary 6.4, this maximum occurs only on the boundary.

Now assume that q is boundary far. Suppose $\omega_d(q) = d(q)$. Because q is boundary far and $b > 0$, for all $z \in \partial(U \setminus \{p\})$ we have $d(q) \geq d(z)$. That is,

$$\omega_d(q) \geq \omega_d(z)$$

for all $z \in \partial(U \setminus \{p\})$. This contradicts the fact that the maximum of ω_d occurs only on the boundary. We conclude that $\omega_d(q) < d(q)$.

The contrapositive of the second assertion is an observation in Section 1.4 of [**1**]. □

Ideally, we would like to prove both converse implications of the above theorem. This, however, is not possible, since if both converse statements are true, we would have proved that all geodesically far points are boundary far, which is not necessarily the case in an arbitrary sub-Riemannian space. For example, the unit ball in the Heisenberg group has points that are geodesically far with respect to the center, but these points are not boundary far. We conclude that the converse statements are not necessarily both true. We have the following lemma that partially addresses this issue.

Lemma 7.3. *Let $U, p, a, b, d(q)$ and $\omega_d(q)$ be as in Theorem 7.2. Additionally, suppose U has points that are boundary far with respect to p and points that are boundary near with respect to p. Then there exists a point $q \in U$ that is boundary near with $\omega_d(q) < d(q)$. Thus, $\omega_d(q) < d(q)$ does not necessarily imply that q is boundary far.*

Proof. Suppose that $\omega_d(q) < d(q)$ implies q is boundary far. Then the logically equivalent implication that q is boundary near implies $\omega_d(q) = d(q)$ would be true. We will show, however, that the latter implication is false.

By the continuity of the distance function, we may construct a sequence $\{q_n\}_{n \in \mathbf{N}}$ of points in U that are boundary near with respect to p and converge to the point $q \in U$ that is boundary far with respect to p. By our assumption, we have $\omega_d(q_n) = d(q_n)$. By continuity of the cone function, this implies $\omega_d(q) = d(q)$. However, q is boundary far, and so Theorem 7.2, which showed that $\omega_d(q) < d(q)$, is contradicted. □

It is an open problem as to whether this geometric condition completely characterizes all domains where $\omega_d(q) < d(q)$ does not necessarily imply that q is boundary far. It is also an open problem to completely characterize the conditions under which $\omega_d(q) = d(q)$ implies q is geodesically near with respect to p and completely characterize the conditions under which $\omega_d(q) < d(q)$ implies q is boundary far with respect to p.

8. Regularity. As in the Euclidean environment, the technique of comparison with cones can be used to prove regularity results. In Lemma 4.2, we addressed the issue of a viscosity infinite harmonic function u being locally Lipschitz, but did not have results for viscosity infinite subharmonic functions and viscosity infinite superharmonic functions. Using Proposition 5.4, we conclude that viscosity infinite subharmonic and superharmonic functions are locally Lipschitz. We may also expand Jensen's [**11**] proof of interior regularity to give us a bound for the Lipschitz constant, namely,

Lemma 8.1. *Let u be a viscosity infinite subharmonic function in Ω. Then u is locally Lipschitz and for almost every $p \in \Omega$, we have the bound*

$$\|\nabla_0 u(p)\| \leq \frac{CM}{d_C(p, \partial\Omega)}$$

where $M = \|u\|_{L^\infty(\Omega)}$ and C is a constant independent of u.

The following corollary has a proof based on the fact that u is a viscosity infinite subharmonic function exactly when $-u$ is a viscosity infinite superharmonic function.

Corollary 8.2. *Let u be a viscosity infinite superharmonic function in Ω. Then u is locally Lipschitz and for almost every $p \in \Omega$, we have the bound*

$$\|\nabla_0 u(p)\| \leq \frac{CM}{d_C(p, \partial\Omega)}$$

where $M = \|u\|_{L^\infty(\Omega)}$ and C is a constant independent of u.

The fact that viscosity infinite subharmonic functions and viscosity infinite superharmonic functions are locally Lipschitz allows us to state the following result, whose proof is a straightforward extension of Jensen's Euclidean proof [**11**].

Lemma 8.3. *Given $\theta \in C(\partial\Omega)$, there is a unique absolute minimizer of θ into Ω. In addition, the absolute minimizer is infinite harmonic.*

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