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Lipschitz Extensions on Generalized Grushin Spaces

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1. Background and Main Results

In this paper, we look to extend the concept of viscosity solutions to Grushin-type spaces, which are constructed using $\mathbb{R}^n$ but lack a group structure. The first part of this article is dedicated to background material and the establishment of Grushin maximum principles. This allows us to prove comparison principles, including one for viscosity infinite harmonic functions. After doing so, the final section is used to prove that C^1_{sub} absolute minimizers are viscosity infinite harmonic. (For the definitions of C^1_{sub} and C^2_{sub} functions, see Definition 1.) This result is inspired by the work of Capogna and the author in [5], in which absolute minimizers in Carnot groups are shown to be viscosity infinite harmonic and C^1_{sub} minimizers in free vector fields are also shown to be viscosity infinite harmonic. The main nicety of the proof here is that the Rothschild–Stein lifting theorem [18] is not needed, for the Taylor polynomial is directly computable.

We begin by constructing the Grushin-type spaces. We consider $\mathbb{R}^n$ with coordinates $(x_1, x_2, \dots, x_n)$ and the vector fields

$$X_i = \rho_i(x_1, x_2, \dots, x_{i-1}) \frac{\partial}{\partial x_i}$$

for $i = 2, 3, \dots, n$, where $\rho_i(x_1, x_2, \dots, x_{i-1})$ is a (possibly constant) polynomial. We decree that $\rho_1 \equiv 1$ so that

$$X_1 = \frac{\partial}{\partial x_1}.$$

A quick calculation shows that when $i < j$, the Lie bracket is given by

$$X_{ij} \equiv [X_i, X_j] = \rho_i(x_1, x_2, \dots, x_{i-1}) \frac{\partial \rho_j(x_1, x_2, \dots, x_{j-1})}{\partial x_i} \frac{\partial}{\partial x_j}. \tag{1.1}$$

Because the ρ_i are polynomials, at each point there is a finite number of iterations of the Lie bracket such that $\partial/\partial x_i$ has a nonzero coefficient. This is easily seen for X_1 and X_2, and the result is obtained inductively for X_i. (We remark that the number of iterations necessary is a function of the point.) Thus, Hörmander's condition is satisfied by these vector fields. Endowing $\mathbb{R}^n$ with an inner product (singular where the polynomials vanish), so that the X_i are orthonormal, produces a manifold that we shall call g_n. This is the tangent space to a generalized Grushin-type

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space G_n. Points in G_n will also be denoted by $p = (x_1, x_2, \dots, x_n)$, with a fixed point denoted $p_0 = (x_1^0, x_2^0, \dots, x_n^0)$. In addition, we use the notation $p - p_0 = (x_1 - x_1^0, x_2 - x_2^0, \dots, x_n - x_n^0)$ and denote the evaluation $\rho_i(p_0)$ by ρ_i^0.

Even though G_n is not a group, it is a metric space whose natural metric is the Carnot–Carathéodory distance, which is defined for the points p and q as follows:

$$d_C(p,q) = \inf_{\Gamma} \int_0^1 \|\gamma'(t)\|\, dt,$$

where the set Γ is the set of all curves γ such that $\gamma(0) = p$, $\gamma(1) = q$, and $\gamma'(t)$ is in $\operatorname{span}\{\{X_i(\gamma(t))\}_{i=1}^n\}$. By Chow's theorem (see e.g. [3]) any two points can be connected by such a curve, which means $d_C(p,q)$ is an honest metric. Using this metric, we can define a Carnot–Carathéodory ball of radius r centered at a point p_0 by

$$B = B(p_0, r) = \{p \in G_n : d_C(p, p_0) < r\};$$

similarly, we shall denote a bounded domain in G_n by Ω. The Carnot–Carathéodory metric behaves differently when the polynomials $\rho_i(x_1, x_2, \dots, x_{i-1})$ vanish. Fixing a point p_0, consider the n-tuple $r_{p_0} = (r_{p_0}^1, r_{p_0}^2, \dots, r_{p_0}^n)$, where $r_{p_0}^i$ is the minimal length of the Lie bracket iteration required to produce

$$[X_{j_1}, [X_{j_2}, [\cdots [X_{j_{r_{p_0}^i}}, X_i] \cdots](p_0) \neq 0.$$

Note that even though the minimal length is unique, the iteration used to obtain that minimum is not unique. Note also that

$$\rho_i(p_0) \neq 0 \iff r_{p_0}^i = 0.$$

Using Theorem 7.34 from [3], we obtain the local estimate at p_0:

$$d_C(p_0, p) \sim \sum_{i=1}^n |x_i - x_i^0|^{1/(1+r_{p_0}^i)}. \tag{1.2}$$

Given a smooth function f on G_n, we define the horizontal gradient of f as

$$\nabla_0 f(p) = (X_1 f(p), X_2 f(p), \dots, X_n f(p))$$

and the symmetrized second-order (horizontal) derivative matrix by

$$((D^2 f(p))^\star)_{ij} = \frac{1}{2}(X_i X_j f(p) + X_j X_i f(p))$$

for $i, j = 1, 2, \dots, n$.

DEFINITION 1. The function $f \colon G_n \mapsto \mathbb{R}$ is said to be C^1_{sub} if $X_i f$ is continuous for all $i = 1, 2, \dots, n$. Similarly, the function f is C^2_{sub} if $X_i X_j f(p)$ is continuous for all $i, j = 1, 2, \dots, n$.

It should also be noted that, for any open set $\mathcal{O} \subset G_n$, the function f is in the horizontal Sobolev space $W^{1,q}(\mathcal{O})$ if $f, X_1 f, \dots, X_n f$ are in $L^q(\mathcal{O})$. Replacing $L^q(\mathcal{O})$ by $L^q_{\text{loc}}(\mathcal{O})$, the space $W^{1,q}_{\text{loc}}(\mathcal{O})$ is defined similarly. The space $W^{1,q}_0(\mathcal{O})$ is the closure in $W^{1,q}(\mathcal{O})$ of smooth functions with compact support. Locally Lipschitz functions are those functions f such that

$$\|\nabla_0 f\|_{L^\infty_{\text{loc}}} < \infty.$$

(See [11] and [12] for further details.)

Using these derivatives, the class of equations we consider are given by

$$F(p, u(p), \nabla_0 u(p), (D^2 u(p))^\star) = 0,$$

where the continuous function

$$F\colon G_n \times \mathbb{R} \times g_n \times S^n \mapsto \mathbb{R}$$

satisfies

$$F(p, r, \eta, X) \le F(p, s, \eta, Y)$$

when $r \le s$ and $Y \le X$. (That is, F is proper [9].) Recall that S^n is the set of $n \times n$ real symmetric matrices. An example of this type of equation is the quasilinear horizontal q-Laplacian

$$\operatorname{div}(\|\nabla_0 u\|^{q-2} \nabla_0 u) = \sum_{i=1}^{n} X_i(\|\nabla_0 u\|^{q-2} X_i u)$$

for $2 \le q < \infty$. Formally taking the limit as $q \to \infty$ yields the horizontal infinite Laplacian

$$\begin{aligned} \Delta_{0,\infty} f &= \sum_{i,j=1}^{n} X_i f X_j f X_i X_j f \\ &= \langle \nabla_0 f, (D^2 f)^\star \nabla_0 f \rangle. \end{aligned}$$

For a more complete discussion of the q-Laplacian and infinite Laplacian, see [4; 13].

We first define solutions to the equation

$$F(p, u(p), \nabla_0 u(p), (D^2 u(p))^\star) = 0$$

in the viscosity sense. In order to do so, we must define the subelliptic jets. (For a thorough discussion of jets, the interested reader is directed to [9].) Given a function $f\colon G_n \mapsto \mathbb{R}$, it is natural to consider inequalities based on the Taylor expansion.

DEFINITION 2. We fix the point p_0 and let $\mathcal{N}$ be the set of indexes so that

$$k \in \mathcal{N} \iff \rho_k^0 = 0.$$

Then, given a function $u\colon G_n \mapsto \mathbb{R}$, a vector $\sum_{i=1}^n \eta_j X_j = \eta \in g_n$, and an S^n matrix X, the pair $(\eta, X) \in J^{2,+}u(p_0)$ if

$$\begin{aligned} u(p) \le u(p_0) &+ \sum_{j \notin \mathcal{N}} \frac{1}{\rho_j^0}(x_j - x_j^0)\eta_j + \frac{1}{2}\sum_{j \notin \mathcal{N}} \frac{1}{(\rho_j^0)^2}(x_j - x_j^0)^2 X_{jj} \\ &+ \sum_{\substack{i,j \notin \mathcal{N} \\ i<j}} (x_i - x_i^0)(x_j - x_j^0)\left(\frac{1}{\rho_j^0 \rho_i^0} X_{ij} - \frac{1}{2}\frac{1}{(\rho_j^0)^2}\frac{\partial \rho_j}{\partial x_i}(p_0)\eta_j\right) \\ &+ \sum_{k \in \mathcal{N}} \frac{1}{\beta} \sum_{j=1}^{n} (x_k - x_k^0)\frac{2}{\rho_j^0}\left(\frac{\partial \rho_k}{\partial x_j}(p_0)\right)^{-1} X_{jk} + o(d_C(p_0, p)^2), \end{aligned}$$

where β is the number of nonzero terms in the final sum.

Here, we understand that if $\rho_j^0 = 0$ or $(\partial\rho_k/\partial x_j)(p_0) = 0$ then that term in the final sum is zero. The second-order subjet of u at p_0, denoted $J^{2,-}u(p_0)$, is defined by

$$J^{2,-}u(p_0) = -J^{2,+}(-u)(p_0).$$

Using these jets, we can define viscosity solutions to our class of functions.

DEFINITION 3. Let $\mathcal{O}$ be an open set in G_n and let $u\colon \mathcal{O} \mapsto \mathbb{R}$. If u is upper semicontinuous and

$$F(p, u(p), \eta, X) \le 0 \quad \text{for all } \ p \in \mathcal{O}, \quad \text{for all } \ (\eta, X) \in J_{\mathcal{O}}^{2,+}u(p),$$

then u is a *viscosity subsolution* of $F(p, u(p), \nabla_0 u(p), (D^2u(p))^\star) = 0$.

If u is lower semicontinuous and

$$F(p, u(p), \eta, X) \ge 0 \quad \text{for all } \ p \in \mathcal{O}, \quad \text{for all } \ (\eta, X) \in J_{\mathcal{O}}^{2,-}u(p),$$

then u is a *viscosity supersolution* of $F(p, u(p), \nabla_0 u(p), (D^2u(p))^\star) = 0$.

The function u is a *viscosity solution* if it is both a viscosity subsolution and a viscosity supersolution.

In order to use the machinery of [9] to prove comparison principles, a relationship between Euclidean and subelliptic jets must be established. This is accomplished through the following lemma.

MAIN LEMMA. *Let the points $p, p_0 \in \mathbb{R}^n$ be denoted by $p = (x_1, x_2, \dots, x_n)$ and $p_0 = (x_1^0, x_2^0, \dots, x_n^0)$. Let $\eta \in \mathbb{R}^n$ and $X \in S^n$. Also, let $\langle \cdot, \cdot \rangle_{\text{eucl}}$ denote the Euclidean inner product in $\mathbb{R}^n$. Then, define the standard Euclidean superjet, denoted $J_{\text{eucl}}^{2,+}$, by*

$$\begin{aligned} J_{\text{eucl}}^{2,+}u(p_0) = \{(\eta, X) : u(p) \le u(p_0) &+ \langle \eta, p - p_0 \rangle_{\text{eucl}} \\ &+ \tfrac{1}{2}\langle X(p - p_0), p - p_0 \rangle_{\text{eucl}} \\ &+ o(\langle p - p_0, p - p_0 \rangle_{\text{eucl}}) \text{ as } p \to p_0\}. \end{aligned}$$

If $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ is a Euclidean vector and $X = \{X_{ij}\}$ is an S^n matrix such that $(\eta, X) \in \bar{J}_{\text{eucl}}^{2,+}u(p_0)$ then $(\tilde{\eta}, Y) \in \bar{J}^{2,+}u(p_0)$, where the vector $\tilde{\eta}$ is defined by

$$\tilde{\eta} = \sum_{i=1}^n \rho_i^0 \eta_i X_i$$

and the symmetric matrix Y is defined by

$$Y_{ij} = \begin{cases} \rho_i^0 \rho_j^0 X_{ij} + \dfrac{1}{2} \dfrac{\partial \rho_j^0}{\partial x_i} \rho_i^0 \eta_j, & i \le j, \\ Y_{ji}, & i > j. \end{cases}$$

This lemma is the key to proving comparison principles. The first comparison principle involves strictly monotone elliptic equations. Such equations satisfy the following properties:

$$
\begin{aligned}
\sigma(r-s) &\leq F(p,r,\eta,X) - F(p,s,\eta,X),\\
|F(p,r,\eta,X) - F(q,r,\eta,X)| &\leq w_1(d_C(p,q)),\\
|F(p,r,\eta,X) - F(p,r,\eta,Y)| &\leq w_2(\|Y-X\|),\\
|F(p,r,\eta,X) - F(p,r,\nu,X)| &\leq w_3(|\|\eta\| - \|\nu\||),
\end{aligned}
$$

where the constant $\sigma > 0$ and the functions $w_i : [0,\infty] \mapsto [0,\infty]$ satisfy $w_i(0^+) = 0$ for $i = 1,2,3$. The appropriate comparison principle is given next.

THEOREM 1.1. *Let F satisfy the four properties just listed. Let u be an upper semicontinuous subsolution and v a lower semicontinuous supersolution to*

$$F(p, f(p), \nabla_0 f(p), (D^2 f(p))^\star) = 0$$

in a domain Ω such that

$$\limsup_{q\to p} u(q) \leq \liminf_{q\to p} v(q)$$

when $p \in \partial\Omega$, where both sides are neither ∞ nor $-\infty$ simultaneously. Then

$$u(p) \leq v(p)$$

for all $p \in \Omega$.

The second comparison principle involves Jensen's auxiliary functions [13], which are used in the proof of uniqueness for infinite harmonic functions in certain Grushin spaces (see Section 5 for complete details). This function is defined by

$$F_\varepsilon(\eta, X) = \min\{\|\eta\|^2 - \varepsilon^2, -\langle X\eta, \eta\rangle\},$$

where ε is a positive real number.

THEOREM 1.2. *Let u be an upper semicontinuous subsolution and v a lower semicontinuous supersolution to*

$$F_\varepsilon(\nabla_0 f(p), (D^2 f(p))^\star) = 0$$

in a domain Ω such that

$$\limsup_{q\to p} u(q) \leq \liminf_{q\to p} v(q)$$

when $p \in \partial\Omega$, where both sides are neither ∞ nor $-\infty$ simultaneously. Then

$$u(p) \leq v(p)$$

for all $p \in \Omega$.

This comparison principle produces a corollary whose proof is similar to that of the theorem.

COROLLARY 1.3. *Let ε be a positive real number. Then a comparison principle for*

$$H_\varepsilon(\eta, X) = \min\{\varepsilon^2 - \|\eta\|^2, -\langle X\eta, \eta\rangle\}$$

holds as in Theorem 1.2.

We then let $\varepsilon \to 0$ to obtain a comparison principle for infinite harmonic functions, as follows.

THEOREM 1.4. *Let u be an upper semicontinuous subsolution and v a lower semicontinuous supersolution of*

$$\Delta_{0,\infty} u = 0$$

in a domain Ω such that, if $p \in \partial\Omega$, then

$$\limsup_{q \to p} u(q) \le \limsup_{q \to p} v(q),$$

where both sides are neither $-\infty$ nor $+\infty$ simultaneously. Then, for all $p \in \Omega$,

$$u(p) \le v(p).$$

Having shown that viscosity infinite harmonic functions are unique, the relationship between absolute minimizers and viscosity infinite harmonic functions is established. Recall that minimal Lipschitz extensions are Lipschitz functions defined in a set $\Omega \subset G_n$ with the property that, for all Lipschitz w such that $u = w$ on $\partial\Omega$,

$$\|u\|_{W^{1,\infty}(\Omega)} \le \|w\|_{W^{1,\infty}(\Omega)}.$$

In general, minimal Lipschitz extensions are neither smooth nor unique. In [1], Aronsson introduced absolutely minimizing Lipschitz extensions, or absolute minimizers, which have the foregoing property on every subset $\hat{\Omega} \subset \Omega$. In [13], Jensen showed that every Euclidean absolute minimizer is viscosity infinite harmonic, and this proof was simplified by Crandall in [8]. In [5], Capogna and the author prove the same result for Carnot groups, showing also that the result is true in free vector fields if the added hypothesis of C^1_{sub} regularity is added. Using ideas from this paper, the following theorem is proved.

THEOREM 1.5. *If u is a C^1_{sub} absolute minimizer, then u is viscosity infinite harmonic. Hence, it is unique in a certain class of Grushin spaces.*

This paper is divided up as follows. Section 2 is concerned with formulating Taylor's theorem on Grushin-type spaces; Section 3 defines second-order jets on Grushin-type spaces and proves needed properties. Section 4 establishes a Grushin maximum principle, and Section 5 proves various comparison principles. In Section 6, absolute minimizers are shown to be viscosity infinite harmonic.

2. Taylor Polynomials

We begin by formally expressing the Taylor polynomial of a function. The Taylor polynomial will depend upon the base point, changing at the zeros of the various ρ_i.

PROPOSITION 2.1. *Let $f : G_n \mapsto \mathbb{R}$ be a C^2_{sub} function and let p_0 be as before. We define the set $\mathcal{N}$ by*

$$k \in \mathcal{N} \iff \rho_k^0 = 0.$$

Then the Taylor polynomial is given by the following formula:

$$\begin{aligned}
f(p) &= f(p_0) + \sum_{j \notin \mathcal{N}} \frac{1}{\rho_j^0}(x_j - x_j^0) X_j f(p_0) + \frac{1}{2} \sum_{j \notin \mathcal{N}} \frac{1}{(\rho_j^0)^2}(x_j - x_j^0)^2 X_j X_j f(p_0) \\
&\quad + \sum_{\substack{i,j \notin \mathcal{N} \\ i<j}} (x_i - x_i^0)(x_j - x_j^0) \biggl(\frac{1}{\rho_i^0 \rho_j^0} \frac{X_i X_j + X_j X_i}{2} f(p_0) \\
&\qquad\qquad - \frac{1}{2} \frac{1}{(\rho_j^0)^2} \frac{\partial \rho_j}{\partial x_i}(p_0) X_j f(p_0) \biggr) \\
&\quad + \sum_{k \in \mathcal{N}} (x_k - x_k^0) \frac{\partial}{\partial x_k} f(p_0) + o(d_C(p_0, p)^2).
\end{aligned}$$

Proof. Denote the right-hand side of the preceding formula by $P(p)$. Let $r, l \notin \mathcal{N}$. Then we calculate that

$$\begin{aligned}
&X_l P(p) \\
&= \rho_l \biggl(\frac{1}{\rho_l^0} X_l f(p_0) + \frac{1}{(\rho_j^0)^2}(x_l - x_l^0) X_l X_l f(p_0) \biggr) \\
&\quad + \rho_l \sum_{i<l} (x_i - x_i^0) \biggl(\frac{1}{\rho_i^0 \rho_l^0} \frac{X_i X_l + X_l X_i}{2} f(p_0) - \frac{1}{2} \frac{1}{(\rho_l^0)^2} \frac{\partial \rho_l}{\partial x_i}(p_0) X_l f(p_0) \biggr) \\
&\quad + \rho_l \sum_{l<j} (x_j - x_j^0) \biggl(\frac{1}{\rho_j^0 \rho_l^0} \frac{X_j X_l + X_l X_j}{2} f(p_0) - \frac{1}{2} \frac{1}{(\rho_j^0)^2} \frac{\partial \rho_j}{\partial x_l}(p_0) X_j f(p_0) \biggr),
\end{aligned}$$

so that at point p_0 we obtain $X_l P(p_0) = X_l f(p_0)$. Differentiating again, we obtain

$$X_l X_l P(p) = (\rho_l)^2 \frac{1}{(\rho_l^0)^2} X_l X_l f(p_0),$$

so that evaluation at p_0 produces $X_l X_l P(p_0) = X_l X_l f(p_0)$.

In order to compute the other second-order derivatives, we must consider the two cases of $r < l$ and $r > l$. We first consider $r < l$. Then,

$$\begin{aligned}
X_r X_l P(p) &= \rho_r \rho_l \biggl(\frac{1}{\rho_r} \frac{1}{\rho_l} \frac{X_r X_l + X_l X_r}{2} f(p_0) - \frac{1}{2} \frac{1}{(\rho_l^0)^2} \frac{\partial \rho_l}{\partial x_r}(p_0) X_l f(p_0) \biggr) \\
&\qquad + \rho_r \frac{\partial \rho_l}{\partial x_r} \frac{X_l P(p)}{\rho_l},
\end{aligned}$$

so that

$$\begin{aligned}
&X_r X_l P(p_0) \\
&\quad = \rho_r^0 \frac{\partial \rho_l}{\partial x_r}(p_0) \frac{\partial}{\partial x_l} f(p_0) + \frac{X_r X_l + X_l X_r}{2} f(p_0) - \frac{1}{2} \rho_r^0 \frac{\partial \rho_l}{\partial x_r}(p_0) \frac{\partial}{\partial x_l} f(p_0) \\
&\quad = \frac{1}{2}[X_r, X_l] f(p_0) + \frac{1}{2}(X_r X_l + X_l X_r) f(p_0) \\
&\quad = X_r X_l f(p_0).
\end{aligned}$$

In the case $r > l$, we note that $(\partial\rho_l/\partial x_r)(p_0) = 0$ so that

$$X_rX_lP(p) = \rho_r\rho_l\biggl(\frac{1}{\rho_r}\frac{1}{\rho_l}\frac{X_rX_l + X_lX_r}{2}f(p_0) - \frac{1}{2}\frac{1}{(\rho_r^0)^2}\frac{\partial\rho_r}{\partial x_l}(p_0)X_rf(p_0)\biggr);$$

hence we obtain

$$\begin{aligned} X_rX_lP(p_0) &= \frac{1}{2}(X_rX_l + X_lX_r)f(p_0) - \frac{1}{2}\rho_l^0\frac{\partial\rho_r}{\partial x_l}(p_0)\frac{\partial}{\partial x_r}f(p_0) \\ &= \frac{1}{2}(X_rX_l + X_lX_r)f(p_0) - \frac{1}{2}[X_l, X_r] = X_rX_lf(p_0). \end{aligned}$$

We now turn our attention to elements of $\mathcal{N}$. Let $b, c \in \mathcal{N}$. Then,

$$\begin{aligned} X_bP(p) &= \rho_b\frac{\partial}{\partial x_b}f(p_0), \\ X_cX_bP(p) &= \rho_c\frac{\partial}{\partial x_c}(X_bP(p)), \end{aligned}$$

and so

$$X_bP(p_0) = X_cX_bP(p_0) = 0.$$

Using the subindex r as before, we also have

$$\begin{aligned} X_bX_rP(p) &= \rho_b\frac{\partial}{\partial x_b}(X_rP(p)), \\ X_rX_bP(p) &= \rho_r\frac{\partial\rho_b}{\partial x_r}\frac{\partial}{\partial x_b}f(p_0), \end{aligned}$$

resulting in

$$X_bX_rP(p_0) = 0 \quad\text{and}\quad X_rX_bP(p_0) = X_rX_bf(p_0).$$

Thus, $P(p)$ is the Taylor polynomial. By Theorem 4.10 in [3], the error is $o(d_C(p, p_0)^2)$. □

3. Grushin Jets

In order to properly define jets using the first- and second-order derivatives, we must first rewrite the Taylor polynomial using only these derivatives. In particular, the terms involving indexes in the null set $\mathcal{N}$ need to be adjusted. We accomplish this by the following proposition.

PROPOSITION 3.1. *Let $i \in \mathcal{N}$. Then we have*

$$\frac{\partial}{\partial x_i}f(p_0) = \frac{1}{\beta}\sum_{j=1}^{n}\frac{2}{\rho_j^0}\biggl(\frac{\partial\rho_i}{\partial x_j}(p_0)\biggr)^{-1}\frac{1}{2}(X_jX_i + X_iX_j)f(p_0)$$

with the understanding that, if $\rho_j^0 = 0$ or $(\partial\rho_i/\partial x_j)(p_0) = 0$, then the whole term is considered to be zero. Here also, β is the number of nonzero terms in the sum.

Proof. As in the proof of the Taylor formula, it is easy to see that, for any fixed j,

$$X_i X_j f(p_0) = 0$$

and also

$$X_j X_i f(p_0) = \rho_j^0 \frac{\partial \rho_i}{\partial x_j}(p_0) \frac{\partial}{\partial x_i} f(p_0).$$

We thus obtain

$$\frac{1}{2}(X_j X_i + X_i X_j) f(p_0) = \frac{1}{2} \rho_j^0 \frac{\partial \rho_i}{\partial x_j}(p_0) \frac{\partial}{\partial x_i} f(p_0).$$

The proposition then follows, with the observation that if either $\rho_j^0 = 0$ or $(\partial \rho_i / \partial x_j)(p_0) = 0$ then the left-hand side of the final equation is also zero. $\square$

With this adjustment, we can now define jets on G_n.

DEFINITION 4. Given a function $u\colon \mathcal{O} \subset G_n \mapsto \mathbb{R}$, a point $p_0 \in \mathcal{O}$, a vector $\sum_{i=1}^n \eta_j X_j = \eta \in g_n$, and an S^n matrix X, the pair $(\eta, X) \in J_{\mathcal{O}}^{2,+} u(p_0)$ if

$$\begin{aligned} u(p) \le u(p_0) &+ \sum_{j \notin \mathcal{N}} \frac{1}{\rho_j^0}(x_j - x_j^0)\eta_j + \frac{1}{2} \sum_{j \notin \mathcal{N}} \frac{1}{(\rho_j^0)^2}(x_j - x_j^0)^2 X_{jj} \\ &+ \sum_{\substack{i,j \notin \mathcal{N} \\ i<j}} (x_i - x_i^0)(x_j - x_j^0)\left(\frac{1}{\rho_j^0 \rho_i^0} X_{ij} - \frac{1}{2} \frac{1}{(\rho_j^0)^2} \frac{\partial \rho_j}{\partial x_i}(p_0)\eta_j \right) \\ &+ \sum_{k \in \mathcal{N}} \frac{1}{\beta} \sum_{j=1}^n (x_k - x_k^0) \frac{2}{\rho_j^0} \left(\frac{\partial \rho_k}{\partial x_j}(p_0) \right)^{-1} X_{jk} + o(d_C(p_0, p)^2). \end{aligned} \tag{3.1}$$

Here again, we understand that if $\rho_j^0 = 0$ or $(\partial \rho_{i_m} / \partial x_j)(p_0) = 0$ then that term in the final sum is zero. We define the subjet $J^{2,-} u(p_0)$ by

$$J^{2,-} u(p_0) = -J^{2,+}(-u)(p_0).$$

Using these jets, we can define viscosity solutions to our class of functions.

DEFINITION 5. Let $\mathcal{O}$ be an open set in G_n and let $u\colon \mathcal{O} \mapsto \mathbb{R}$. If u is upper semicontinuous and

$$F(p, u(p), \eta, X) \le 0 \quad \text{for all } \ p \in \mathcal{O}, \quad \text{for all } \ (\eta, X) \in J_{\mathcal{O}}^{2,+} u(p),$$

then u is a *viscosity subsolution* of $F(p, u(p), \nabla_0 u(p), (D^2 u(p))^\star) = 0$.

If u is lower semicontinuous and

$$F(p, u(p), \eta, X) \ge 0 \quad \text{for all } \ p \in \mathcal{O}, \quad \text{for all } \ (\eta, X) \in J_{\mathcal{O}}^{2,-} u(p),$$

then u is a *viscosity supersolution* of $F(p, u(p), \nabla_0 u(p), (D^2 u(p))^\star) = 0$.

The function u is a *viscosity solution* if it is both a viscosity subsolution and a viscosity supersolution.

The following proposition establishes the correspondence between elements of the superjet and functions that touch from above. The proof is an extension of proofs in [7] and [4], so only key points of the proof will be highlighted.

PROPOSITION 3.2. *Let $u\colon G_n \mapsto \mathbb{R}$ and let $\mathcal{A}_{p_0}$ be the set of C^2_{sub} functions such that*

$$\phi \in \mathcal{A}_{p_0} \iff u - \phi \text{ has a local max at } p_0.$$

Then we have the characterization of the superjets by

$$J^{2,+}u(p_0) = \{(\nabla_0\phi(p_0), (D^2\phi(p_0))^\star) : \phi \in \mathcal{A}_{p_0}\}.$$

Proof. The containment

$$\{(\nabla_0\phi(p_0), (D^2\phi(p_0))^\star) : \phi \in \mathcal{A}_{p_0}\} \subset J^{2,+}u(p_0)$$

follows easily from the Taylor polynomial. To show the other direction, we define the function $a\colon G_n \mapsto \mathbb{R}$ by

$$a(p) = \sum_{i=1}^n x_i^4.$$

Then $a(p-p_0)$ is C^2_{sub} and $O(d_C(p,p_0)^4)$. Assuming $(\eta, X) \in J^{2,+}u(p_0)$, we denote the non-error part of the right-hand side of inequality (3.1) by $\mathcal{P}(p)$. Define the function $z(s)$ by

$$z(s) = \sup\{(u(p) - \mathcal{P}(p))^+\},$$

where the sup is taken over the region $d_C(p,p_0) \leq s$. Then the function

$$\phi(p) = \zeta(d_C(p,p_0)) + a(p-p_0) + \mathcal{P}(p),$$

with ζ constructed as in [7], has the desired properties. □

In addition to having properties similar to those of Euclidean jets, our jets are related to Euclidean jets through the following twisting lemma. This main lemma will enable us to prove comparison principles.

MAIN LEMMA. *Let the points $p, p_0 \in \mathbb{R}^n$ be denoted by $p = (x_1, x_2, \ldots, x_n)$ and $p_0 = (x_1^0, x_2^0, \ldots, x_n^0)$. Let $\eta \in \mathbb{R}^n$ and $X \in S^n$. Also, let $\langle\cdot,\cdot\rangle_{\text{eucl}}$ denote the Euclidean inner product in $\mathbb{R}^n$. Then, define the standard Euclidean superjet, denoted $J^{2,+}_{\text{eucl}}$, by*

$$\begin{aligned} J^{2,+}_{\text{eucl}}u(p_0) = \{(\eta, X) : u(p) \leq u(p_0) &+ \langle \eta, p - p_0\rangle_{\text{eucl}} \\ &+ \tfrac{1}{2}\langle X(p-p_0), p-p_0\rangle_{\text{eucl}} \\ &+ o(\langle p-p_0, p-p_0\rangle_{\text{eucl}}) \text{ as } p \to p_0\}. \end{aligned}$$

If $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$ is a Euclidean vector and $X = \{X_{ij}\}$ is an S^n matrix such that $(\eta, X) \in \bar{J}^{2,+}_{\text{eucl}}u(p_0)$ then $(\tilde{\eta}, Y) \in \bar{J}^{2,+}u(p_0)$, where the vector $\tilde{\eta}$ is defined by

$$\tilde{\eta} = \sum_{i=1}^n \rho_i^0 \eta_i X_i$$

and the symmetric matrix Y is defined by

$$Y_{ij} = \begin{cases} \rho_i^0 \rho_j^0 X_{ij} + \dfrac{1}{2}\dfrac{\partial \rho_j^0}{\partial x_i}\rho_i^0 \eta_j, & i \le j, \\ Y_{ji}, & i > j. \end{cases}$$

Proof. Using the estimate (1.2) yields

$$h = o(|p - p_0|^2) \implies h = o(d_C(p, p_0)^2)$$

and so, recalling that when $k \in \mathcal{N}$ we have $\rho_k^0 = 0$, we obtain

$$\begin{aligned} u(p) &\le u(p_0) + \sum_{i=1}^n \eta_i (x_i - x_i^0) + \frac{1}{2}\langle X(p - p_0), p - p_0\rangle + o(|p - p_0|^2) \\ &= u(p_0) + \sum_{i \notin \mathcal{N}} \frac{1}{\rho_i^0}\tilde{\eta}_i (x_i - x_i^0) + \sum_{j \in \mathcal{N}} \eta_j (x_j - y_j) \\ &\quad + \frac{1}{2}\sum_{l,m=1}^n X_{lm}(x_l - x_l^0)(x_m - x_m^0) + o(d_C(p, p_0)^2) \\ &= u(p_0) + \sum_{i \notin \mathcal{N}} \frac{1}{\rho_i^0}\tilde{\eta}_i (x_i - x_i^0) + \frac{1}{2}\sum_{i \notin \mathcal{N}} \frac{1}{(\rho_i^0)^2} Y_{ii}(x_i - x_i^0)^2 \\ &\quad + \sum_{\substack{i,j \notin \mathcal{N} \\ i<j}} \left(\frac{1}{\rho_i^0 \rho_j^0} Y_{ij} - \frac{1}{2}\frac{\partial \rho_j}{\partial x_i}(p_0)\frac{1}{(\rho_j^0)^2}\tilde{\eta}_j \right)(x_i - x_i^0)(x_j - x_j^0) \\ &\quad + \sum_{j \in \mathcal{N}} \frac{1}{\beta}\sum_{k=1}^n (x_j - x_j^0)\frac{2}{\rho_k^0}\left(\frac{\partial \rho_j}{\partial x_k}(p_0)\right)^{-1} Y_{kj} + o(d_C(p, p_0)^2). \end{aligned}$$

And so the case when $(\eta, X) \in J^{2,+}_{\text{eucl}} u(p_0)$ follows. Otherwise, there is a sequence $\{p_i, \eta_i, X_i\} \in \mathbb{R}^n \times \mathbb{R}^n \times S^n$ such that $(\eta_i, X_i) \in J^{2,+}_{\text{eucl}} u(p_i)$ and $\{p_i, u(p_i), \eta_i, X_i\} \to (p_0, u(p_0), \eta, X)$. Now, (η_i, X_i) can be identified with $(\tilde{\eta}_i, Y_i) \in J^{2,+}u(p_i)$. By construction, $\tilde{\eta}_i \to \tilde{\eta}$ and $Y_i \to Y$. Hence, $(p_i, u(p_i), \tilde{\eta}_i, Y_i) \to (p_0, u(p_0), \tilde{\eta}, Y)$ and so we have $(\tilde{\eta}, Y) \in \bar{J}^{2,+}u(p_0)$. □

4. Maximum Principles

We begin by stating a lemma analogous to Lemma 3.1 of [9]. The proof is similar and thus is omitted.

LEMMA 4.1. *Let u be an upper semicontinuous function in Ω and v a lower semicontinuous function in Ω. For $\tau > 0$ and for points p and q given by $p = (x_1, x_2, \ldots, x_n)$ and $q = (y_1, y_2, \ldots, y_n)$, let the function $\varphi(p, q)$ be defined by*

$$\varphi(p, q) \equiv \sum_{i=1}^n \frac{1}{2^i}(x_i - y_i)^{2^i}$$

and let the function M_τ be defined by

$$M_\tau = \sup_{\bar{\Omega}\times\bar{\Omega}}(u(p) - v(q) - \tau\varphi(p,q)).$$

Let $p_\tau = (x_1^\tau, x_2^\tau, \ldots, x_n^\tau)$ and $q_\tau = (y_1^\tau, y_2^\tau, \ldots, y_n^\tau)$ be such that

$$\lim_{\tau\to\infty}(M_\tau - (u(p_\tau) - v(q_\tau) - \tau\varphi(p_\tau, q_\tau))) = 0.$$

Then,

$$\lim_{\tau\to\infty} \tau\varphi(p_\tau, q_\tau) = 0 \tag{4.1}$$

and

$$\lim_{\tau\to\infty} M_\tau = u(p^*) - v(p^*) = \sup_{\bar{\Omega}}(u(p) - v(p)) \tag{4.2}$$

whenever p^* is a limit point of p_τ as $\tau \mapsto \infty$.

Using the function $\varphi(p_\tau, q_\tau)$, we compute some important vectors and matrices that are dependent upon the Euclidean derivatives. We begin by defining the vectors Υ_{p_τ} and Υ_{q_τ} by

$$\begin{aligned}
(\Upsilon_{p_\tau})_i &\equiv \rho_i(p_\tau)\frac{\partial\varphi(p_\tau, q_\tau)}{\partial x_i} = \rho_i(p_\tau)(x_i^\tau - y_i^\tau)^{2^i-1},\\
(\Upsilon_{q_\tau})_i &\equiv -\rho_i(q_\tau)\frac{\partial\varphi(p_\tau, q_\tau)}{\partial y_i} = \rho_i(q_\tau)(x_i^\tau - y_i^\tau)^{2^i-1}.
\end{aligned}$$

Note that Υ_{p_τ} is the Euclidean derivative of $\varphi(p_\tau, q_\tau)$ with respect to p_τ twisted at p_τ using the Main Lemma and that Υ_{q_τ} is likewise the negative of the Euclidean derivative of $\varphi(p_\tau, q_\tau)$ with respect to q_τ twisted at q_τ using the Main Lemma. Next, we consider the matrix $D^2\varphi(p_\tau, q_\tau)$ of second-order Euclidean derivatives. A straightforward computation shows that $(D^2\varphi(p_\tau, q_\tau))^2 + D^2\varphi(p_\tau, q_\tau)$, which we shall denote by $\mathcal{C}$, has the block form

$$\begin{pmatrix} B & -B \\ -B & B \end{pmatrix},$$

where the $n \times n$ matrix B has elements

$$B_{ij} = \begin{cases} 2(2^i - 1)^2(x_i^\tau - y_i^\tau)^{2^{i+1}-4} + (2^i - 1)(x_i^\tau - y_i^\tau)^{2^i-2}, & i = j, \\ 0, & i \neq j. \end{cases}$$

We now proceed as in [9]. Let u be a viscosity subsolution and v a viscosity supersolution to $F(p, f(p), \nabla_0 f(p), (D^2f(p))^\star) = 0$. Denote the points p and q by $p = (x_1, x_2, \ldots, x_n)$ and $q = (y_1, y_2, \ldots, y_n)$, and let $(p_\tau, q_\tau) = ((x_1^\tau, \ldots, x_n^\tau), (y_1^\tau, \ldots, y_n^\tau))$ be the maximum point of

$$u(p) - v(q) - \tau\varphi(p,q)$$

in $\bar{\Omega} \times \bar{\Omega}$. By the Euclidean maximum principle of semicontinuous functions [9], there exist subsequences $p_{\tau_i} \to p_0$ and $q_{\tau_i} \to p_0$. Passing to the subsequence, we shall denote these points by p_τ and q_τ, respectively. In addition, there exist S^n matrices $X^\tau = \{X_{ij}\}$ and $Y^\tau = \{Y_{ij}\}$ such that

$$(\tau D_p\varphi(p_\tau, q_\tau), X^\tau) \in \bar{J}^{2,+}_{\mathrm{eucl}} u(p_\tau) \quad \text{and} \quad (-\tau D_q\varphi(p_\tau, q_\tau), Y^\tau) \in \bar{J}^{2,-}_{\mathrm{eucl}} v(q_\tau).$$

In addition, the matrices X^τ and Y^τ satisfy the estimate

$$\langle X^\tau \varepsilon, \varepsilon\rangle_{\text{eucl}} - \langle Y^\tau \kappa, \kappa\rangle_{\text{eucl}} \leq \tau\langle \mathcal{C}\chi, \chi\rangle_{\text{eucl}} \tag{4.3}$$

for any vectors ε and κ in $\mathbb{R}^n$, where $\langle\cdot,\cdot\rangle_{\text{eucl}}$ is the standard Euclidean inner product and the vector $\chi = (\varepsilon, \kappa)$. Using the Main Lemma, we obtain

$$(\tau\Upsilon_{p_\tau}, \mathcal{X}^\tau) \in \bar{J}^{2,+}u(p_\tau) \quad \text{and} \quad (\tau\Upsilon_{q_\tau}, \mathcal{Y}^\tau) \in \bar{J}^{2,-}v(q_\tau),$$

where the matrices $\mathcal{X}^\tau$ and $\mathcal{Y}^\tau$ are defined using the Main Lemma—namely, by

$$\mathcal{X}^\tau_{ij} = \begin{cases} \rho_i(p_\tau)\rho_j(p_\tau)X_{ij} + \dfrac{1}{2}\dfrac{\partial\rho_j}{\partial x_i}(p_\tau)\rho_i(p_\tau)\tau(x^\tau_j - y^\tau_j)^{2^j-1}, & i \leq j, \\ \mathcal{X}^\tau_{ji}, & i > j, \end{cases}$$

and by

$$\mathcal{Y}^\tau_{ij} = \begin{cases} \rho_i(q_\tau)\rho_j(q_\tau)Y_{ij} + \dfrac{1}{2}\dfrac{\partial\rho_j}{\partial x_i}(q_\tau)\rho_i(q_\tau)\tau(x^\tau_j - y^\tau_j)^{2^j-1}, & i \leq j, \\ \mathcal{Y}^\tau_{ji}, & i > j. \end{cases}$$

These elements of the subelliptic jets also satisfy important estimates, as given by the following lemma.

LEMMA 4.2. *The vectors Υ_{p_τ} and Υ_{q_τ} satisfy*

$$|\|\Upsilon_{q_\tau}\|^2 - \|\Upsilon_{p_\tau}\|^2| = O(\varphi(p_\tau, q_\tau)^2). \tag{4.4}$$

In addition, with the usual ordering, the matrix $\mathcal{X}^\tau$ is smaller than the matrix $\mathcal{Y}^\tau$ with an error term. In particular we have $\mathcal{X}^\tau \leq \mathcal{Y}^\tau + \mathcal{R}^\tau$, where $\mathcal{R}^\tau \to 0$ as $\tau \to \infty$.

Proof. A straightforward computation shows

$$\|\Upsilon_{q_\tau}\|^2 - \|\Upsilon_{p_\tau}\|^2 = \sum_{i=1}^n ((\rho_i(p_\tau))^2 - (\rho_i(q_\tau))^2)(x_i - y_i)^{2^{i+1}-2}.$$

The definition of φ and ρ_i gives the first term as $O(\varphi^{1/(2^i-1)})$, and the second term is clearly $O(\varphi^{(2^{i+1}-2)/2^i})$; the vector difference estimate follows.

We now focus on the matrix difference estimate. We recall the notation from the Main Lemma, in particular, the twisted vector $\tilde{v}$. To emphasize the point at which the functions ρ_i are evaluated, we denote the vector by $\tilde{v}_p$ for evaluation at p_τ and with an analogous definition for $\tilde{v}_q$. Using the definitions of $\mathcal{X}^\tau$ and $\mathcal{Y}^\tau$, we have

$$\begin{aligned} &\langle \mathcal{X}^\tau\varepsilon, \varepsilon\rangle - \langle \mathcal{Y}^\tau\kappa, \kappa\rangle \\ &\quad = \langle X^\tau\tilde{\varepsilon}_p, \tilde{\varepsilon}_p\rangle - \langle Y^\tau\tilde{\kappa}_q, \tilde{\kappa}_q\rangle \\ &\quad\quad + \sum_{j=1}^n\sum_{i\leq j}\tau(x^\tau_j - y^\tau_j)^{2^j-1}\left(\left(\frac{\partial\rho_j}{\partial x_i}\rho_i\right)(p_\tau)\varepsilon_i\varepsilon_j - \left(\frac{\partial\rho_j}{\partial x_i}\rho_i\right)(q_\tau)\kappa_i\kappa_j\right) \quad (4.5) \\ &\quad \leq \tau\langle\mathcal{C}\xi, \xi\rangle \\ &\quad\quad + \sum_{j=1}^n\sum_{i\leq j}\tau(x^\tau_j - y^\tau_j)^{2^j-1}\left(\left(\frac{\partial\rho_j}{\partial x_i}\rho_i\right)(p_\tau)\varepsilon_i\varepsilon_j - \left(\frac{\partial\rho_j}{\partial x_i}\rho_i\right)(q_\tau)\kappa_i\kappa_j\right), \quad (4.6) \end{aligned}$$

where the vector $\xi = (\tilde{\varepsilon}_p, \tilde{\kappa}_q)$.

By the foregoing computation of $\mathcal{C}$ we obtain

$$\langle \mathcal{C}\xi, \xi\rangle = \sum_{i=1}^{n} (\rho_i(p_\tau)\varepsilon_i - \rho_i(q_\tau)\kappa_i)^2 \times \big(2(2^i-1)^2(x_i^\tau - y_i^\tau)^{2^{i+1}-4} + (2^i-1)(x_i^\tau - y_i^\tau)^{2^i-2}\big), \quad (4.7)$$

so that if $\varepsilon = \kappa$ then the term corresponding to $i = 1$ is 0; hence

$$\langle \mathcal{C}\xi, \xi\rangle \sim \sum_{i=2}^{n} \varphi^{2/(2^{i-1})}\varphi^{(2^i-2)/2^i}$$

and so

$$\lim_{\tau\to\infty} \tau\langle \mathcal{C}\xi, \xi\rangle = 0.$$

We now focus on the polynomial term. Observe that, as in the matrix difference, we have

$$\left(\frac{\partial\rho_j}{\partial x_i}\rho_i\right)(p_\tau) - \left(\frac{\partial\rho_j}{\partial x_i}\rho_i\right)(q_\tau) \sim \varphi^{1/(2^j-1)};$$

therefore, if $\varepsilon = \kappa$ then

$$\lim_{\tau\to\infty} \sum_{j=1}^{n}\sum_{i\le j} \tau(x_j^\tau - y_j^\tau)^{2^j-1}\left(\left(\frac{\partial\rho_j}{\partial x_i}\rho_i\right)(p_\tau) - \left(\frac{\partial\rho_j}{\partial x_i}\rho_i\right)(q_\tau)\right) = 0.$$

We thus conclude that $\mathcal{X}^\tau \le \mathcal{Y}^\tau + \mathcal{R}^\tau$ with $\mathcal{R}^\tau \to 0$ as $\tau \to \infty$. $\square$

In order to study infinite harmonic functions, we need to extend Lemma 4.1 to the multivariate case. This extension is given by the next lemma.

LEMMA 4.3. *Let u, v, and Ω be as in Lemma 4.1. Let*

$$\sup_{\bar\Omega}(u(p) - v(p)) = u(p_0) - v(p_0) > 0$$

occur in the interior of Ω. Consider a real vector $\vec\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with nonnegative components as well as the points p and q with coordinates $p = (x_1, x_2, \ldots, x_n)$ and $q = (y_1, y_2, \ldots, y_n)$. We define the following functions:

$$\varphi_{\alpha_1,\alpha_2,\ldots,\alpha_n}(p,q) = \sum_{i=1}^{n} \frac{1}{2}\alpha_i(x_i - y_i)^2,$$

$$\varphi_{\alpha_2,\alpha_3,\ldots,\alpha_n}(p,q) = \sum_{i=2}^{n} \frac{1}{2}\alpha_i(x_i - y_i)^2,$$

$$\varphi_{\alpha_3,\alpha_4,\ldots,\alpha_n}(p,q) = \sum_{i=3}^{n} \frac{1}{2}\alpha_i(x_i - y_i)^2,$$

$$\vdots$$

$$\varphi_{\alpha_n} = \frac{1}{2}\alpha_n(x_n - y_n)^2.$$

Using these functions and upper semicontinuity on a compact set, we can consider the following functions:

$$\begin{aligned}
M_{\alpha_1,\alpha_2,\ldots,\alpha_n} &= \sup_{\bar\Omega\times\bar\Omega}(u(p)-v(q)-\varphi_{\alpha_1,\alpha_2,\ldots,\alpha_n}(p,q))\\
&= u(p_{\alpha_1,\alpha_2,\ldots,\alpha_n})-v(q_{\alpha_1,\alpha_2,\ldots,\alpha_n})\\
&\quad -\varphi_{\alpha_1,\alpha_2,\ldots,\alpha_n}(p_{\alpha_1,\alpha_2,\ldots,\alpha_n},q_{\alpha_1,\alpha_2,\ldots,\alpha_n}),\\
M_{\alpha_2,\alpha_3,\ldots,\alpha_n} &= \sup_{\bar\Omega\times\bar\Omega}(u(p)-v(q)-\varphi_{\alpha_2,\alpha_3,\ldots,\alpha_n}(p,q):x_1=y_1)\\
&= u(p_{\alpha_2,\alpha_3,\ldots,\alpha_n})-v(q_{\alpha_2,\alpha_3,\ldots,\alpha_n})\\
&\quad -\varphi_{\alpha_2,\alpha_3,\ldots,\alpha_n}(p_{\alpha_2,\alpha_3,\ldots,\alpha_n},q_{\alpha_2,\alpha_3,\ldots,\alpha_n}),\\
M_{\alpha_3,\alpha_4,\ldots,\alpha_n} &= \sup_{\bar\Omega\times\bar\Omega}(u(p)-v(q)-\varphi_{\alpha_1,\alpha_2,\ldots,\alpha_n}(p,q):x_1=y_1,\ x_2=y_2)\\
&= u(p_{\alpha_3,\alpha_4,\ldots,\alpha_n})-v(q_{\alpha_3,\alpha_4,\ldots,\alpha_n})\\
&\quad -\varphi_{\alpha_3,\alpha_4,\ldots,\alpha_n}(p_{\alpha_3,\alpha_4,\ldots,\alpha_n},q_{\alpha_3,\alpha_4,\ldots,\alpha_n}),\\
&\ \ \vdots\\
M_{\alpha_n} &= \sup_{\bar\Omega\times\bar\Omega}(u(p)-v(q)-\varphi_{\alpha_n}(p,q):x_i=y_i,\ i=1,2,\ldots,n-1)\\
&= u(p_{\alpha_n})-v(q_{\alpha_n})-\varphi_{\alpha_n}(p_{\alpha_n},q_{\alpha_n}).
\end{aligned}$$

We then have

$$\lim_{\alpha_n\to\infty}\lim_{\alpha_{n-1}\to\infty}\cdots\lim_{\alpha_2\to\infty}\lim_{\alpha_1\to\infty}M_{\alpha_1,\alpha_2,\ldots,\alpha_n}=u(p_0)-v(p_0)$$

and

$$\lim_{\alpha_n\to\infty}\lim_{\alpha_{n-1}\to\infty}\cdots\lim_{\alpha_2\to\infty}\lim_{\alpha_1\to\infty}\varphi_{\alpha_1,\alpha_2,\ldots,\alpha_n}(p_{\alpha_1,\alpha_2,\ldots,\alpha_n},q_{\alpha_1,\alpha_2,\ldots,\alpha_n})=0.$$

Note that as a consequence, for each i, $p_{\alpha_i,\alpha_{i+1},\ldots,\alpha_n}$ and $q_{\alpha_i,\alpha_{i+1},\ldots,\alpha_n}$ have their first $i-1$ coordinates equal. That is, for $j=1$ to $i-1$,

$$x_j^{\alpha_i,\alpha_{i+1},\ldots,\alpha_n}=y_j^{\alpha_i,\alpha_{i+1},\ldots,\alpha_n}.$$

Proof. Begin with α_1. Since we are taking iterated limits, we hold the other coordinates fixed. Then, as in the one-dimensional case [9], we have that

$$\lim_{\alpha_1\to\infty}M_{\alpha_1,\alpha_2,\ldots,\alpha_n}$$

exists because it is decreasing as $\alpha_1\to\infty$ and

$$M_{\alpha_1,\alpha_2,\ldots,\alpha_n}\geq u(p_0)-v(p_0)>0. \tag{4.8}$$

Thus, it is also finite. We then have

$$\begin{aligned}
M_{(1/2)\alpha_1,\alpha_2,\ldots,\alpha_n} &\geq u(p_{\alpha_1,\alpha_2,\ldots,\alpha_n})-v(q_{\alpha_1,\alpha_2,\ldots,\alpha_n})\\
&\quad -\varphi_{\alpha_1,\alpha_2,\ldots,\alpha_n}(p_{\alpha_1,\alpha_2,\ldots,\alpha_n},q_{\alpha_1,\alpha_2,\ldots,\alpha_n})\\
&\quad +\tfrac14\alpha_1(x_1^{\alpha_1,\alpha_2,\ldots,\alpha_n}-y_1^{\alpha_1,\alpha_2,\ldots,\alpha_n})^2\\
&= M_{\alpha_1,\alpha_2,\ldots,\alpha_n}+\tfrac14\alpha_1(x_1^{\alpha_1,\alpha_2,\ldots,\alpha_n}-y_1^{\alpha_1,\alpha_2,\ldots,\alpha_n})^2.
\end{aligned}$$

We thus conclude that

$$\lim_{\alpha_1\to\infty}\alpha_1(x_1^{\alpha_1,\alpha_2,\ldots,\alpha_n}-x_2^{\alpha_1,\alpha_2,\ldots,\alpha_n})^2=0.$$

In addition,

$$\begin{aligned} M_{\alpha_2,\alpha_3,\ldots,\alpha_n} &\le M_{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n} \\ &= u(p_{\alpha_1,\alpha_2,\ldots,\alpha_n}) - v(q_{\alpha_1,\alpha_2,\ldots,\alpha_n}) \\ &\quad - \varphi_{\alpha_1,\alpha_2,\ldots,\alpha_n}(p_{\alpha_1,\alpha_2,\ldots,\alpha_n}, q_{\alpha_1,\alpha_2,\ldots,\alpha_n}), \end{aligned}$$

so that

$$\begin{aligned} M_{\alpha_2,\alpha_3,\ldots,\alpha_n} &\le \lim_{\alpha_1\to\infty} M_{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n} \\ &= \lim_{\alpha_1\to\infty} \big(u(p_{\alpha_1,\alpha_2,\ldots,\alpha_n}) - v(q_{\alpha_1,\alpha_2,\ldots,\alpha_n}) \\ &\qquad - \varphi_{\alpha_1,\alpha_2,\ldots,\alpha_n}(p_{\alpha_1,\alpha_2,\ldots,\alpha_n}, q_{\alpha_1,\alpha_2,\ldots,\alpha_n}) \big) \\ &= u(p^\star) - v(q^\star) - \varphi_{\alpha_2,\alpha_3,\ldots,\alpha_n}(p^\star, q^\star) \le M_{\alpha_2,\alpha_3,\ldots,\alpha_n}, \end{aligned}$$

where

$$p^\star = \lim_{\alpha_1\to\infty} p_{\alpha_1,\alpha_2,\ldots,\alpha_n} \quad \text{and} \quad q^\star = \lim_{\alpha_1\to\infty} q_{\alpha_1,\alpha_2,\ldots,\alpha_n}.$$

Therefore,

$$\begin{aligned} M_{\alpha_2,\alpha_3,\ldots,\alpha_n} &= \lim_{\alpha_1\to\infty} M_{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n}, \\ \lim_{\alpha_1\to\infty} p_{\alpha_1,\alpha_2,\ldots,\alpha_n} &= p_{\alpha_2,\alpha_3,\ldots,\alpha_n}, \\ \lim_{\alpha_1\to\infty} q_{\alpha_1,\alpha_2,\ldots,\alpha_n} &= q_{\alpha_2,\alpha_3,\ldots,\alpha_n}, \end{aligned}$$

and we observe that $p_{\alpha_2,\alpha_3,\ldots,\alpha_n}$ and $q_{\alpha_2,\alpha_3,\ldots,\alpha_n}$ have the same first coordinate; that is, $x_1^{\alpha_2,\alpha_3,\ldots,\alpha_n} = y_1^{\alpha_2,\alpha_3,\ldots,\alpha_n}$.

We then proceed by fixing α_2. As before, $\lim_{\alpha_2\to\infty} M_{\alpha_2,\alpha_3,\ldots,\alpha_n}$ exists and

$$M_{(1/2)\alpha_2,\alpha_3,\ldots,\alpha_n} \ge M_{\alpha_2,\alpha_3,\ldots,\alpha_n} + \tfrac{1}{4}\alpha_2 (x_2^{\alpha_2,\ldots,\alpha_n} - y_2^{\alpha_2,\ldots,\alpha_n})^2,$$

so that we also conclude

$$\lim_{\alpha_2\to\infty} \alpha_2 (x_2^{\alpha_2,\ldots,\alpha_n} - y_2^{\alpha_2,\ldots,\alpha_n})^2 = 0.$$

We then obtain

$$\begin{aligned} &M_{\alpha_3,\alpha_4,\ldots,\alpha_n} \\ &\quad \le M_{\alpha_2,\alpha_3,\ldots,\alpha_n} \\ &\quad = u(p_{\alpha_2,\alpha_3,\ldots,\alpha_n}) - v(q_{\alpha_2,\alpha_3,\ldots,\alpha_n}) - \varphi_{\alpha_2,\alpha_3,\ldots,\alpha_n}(p_{\alpha_2,\alpha_3,\ldots,\alpha_n}, q_{\alpha_2,\alpha_3,\ldots,\alpha_n}) \end{aligned}$$

and hence

$$\begin{aligned} M_{\alpha_3,\alpha_4,\ldots,\alpha_n} &\le \lim_{\alpha_2\to\infty} M_{\alpha_2,\alpha_3,\ldots,\alpha_n} \\ &= \lim_{\alpha_2\to\infty} \big(u(p_{\alpha_2,\alpha_3,\ldots,\alpha_n}) - v(q_{\alpha_2,\alpha_3,\ldots,\alpha_n}) \\ &\qquad - \varphi_{\alpha_2,\alpha_3,\ldots,\alpha_n}(p_{\alpha_2,\alpha_3,\ldots,\alpha_n}, q_{\alpha_2,\alpha_3,\ldots,\alpha_n}) \big) \\ &= u(p^\star) - v(q^\star) - \varphi_{\alpha_3,\alpha_4,\ldots,\alpha_n}(p^\star, q^\star) \le M_{\alpha_3,\alpha_4,\ldots,\alpha_n}, \end{aligned}$$

where

$$p^\star = \lim_{\alpha_2\to\infty} p_{\alpha_2,\alpha_3,\dots,\alpha_n} \quad \text{and} \quad q^\star = \lim_{\alpha_2\to\infty} q_{\alpha_2,\alpha_3,\dots,\alpha_n}.$$

We then have

$$\begin{aligned} M_{\alpha_3,\alpha_4,\dots,\alpha_n} &= \lim_{\alpha_2\to\infty} M_{\alpha_2,\alpha_3,\dots,\alpha_n}, \\ \lim_{\alpha_2\to\infty} p_{\alpha_2,\alpha_3,\dots,\alpha_n} &= p_{\alpha_3,\alpha_4,\dots,\alpha_n}, \\ \lim_{\alpha_2\to\infty} q_{\alpha_2,\alpha_3,\dots,\alpha_n} &= q_{\alpha_3,\alpha_4,\dots,\alpha_n}. \end{aligned}$$

We repeat this process through the n limits, with the last iteration directly following from the one-dimensional version. The lemma then follows. □

The same methodology as used in the proof of Lemma 4.3 yields our next corollary; the details are left to the reader.

COROLLARY 4.4. *Under the hypotheses of Lemma 4.3, it follows that each iterated limit of the form*

$$\lim_{\alpha_{i_1}\to\infty} \lim_{\alpha_{i_2}\to\infty} \lim_{\alpha_{i_3}\to\infty} \cdots \lim_{\alpha_{i_n}\to\infty} M_{\alpha_1,\alpha_2,\dots,\alpha_n}$$

exists and is finite. Thus,

$$\lim_{\alpha_{i_1}\to\infty} \lim_{\alpha_{i_2}\to\infty} \lim_{\alpha_{i_3}\to\infty} \cdots \lim_{\alpha_{i_n}\to\infty} \varphi(p_{\alpha_1,\alpha_2,\dots,\alpha_n}, q_{\alpha_1,\alpha_2,\dots,\alpha_n}) = 0$$

and

$$\lim_{\alpha_{i_1}\to\infty} \lim_{\alpha_{i_2}\to\infty} \lim_{\alpha_{i_3}\to\infty} \cdots \lim_{\alpha_{i_n}\to\infty} M_{\alpha_1,\alpha_2,\dots,\alpha_n} = \sup_{\bar\Omega}(u(p)-v(p)) = u(p_0)-v(p_0).$$

Using Lemma 4.3 and Corollary 4.4, we deduce that every possible iterated limit exists. We do note, however, that the intermediate limit points at which the maxima occur change with each different iteration. It is natural to ask whether the full multivariate limit exists. We answer this in the affirmative via the next lemma.

LEMMA 4.5. *Under the hypotheses of Lemma 4.3, the full limit exists and equals the common value of the iterated limits. That is,*

$$\lim_{\alpha_1,\alpha_2,\dots,\alpha_n\to\infty} M_{\alpha_1,\alpha_2,\dots,\alpha_n} = \sup_{\bar\Omega}(u(p) - v(p)) = u(p_0) - v(p_0). \tag{4.9}$$

As a consequence,

$$\lim_{\alpha_1,\alpha_2,\dots,\alpha_n\to\infty} \varphi_{\alpha_1,\alpha_2,\dots,\alpha_n}(p_{\alpha_1,\alpha_2,\dots,\alpha_n}, q_{\alpha_1,\alpha_2,\dots,\alpha_n}) = 0. \tag{4.10}$$

Proof. Given $\varepsilon > 0$, the fact that

$$\lim_{\alpha_n\to\infty} \lim_{\alpha_{n-1}\to\infty} \cdots \lim_{\alpha_2\to\infty} \lim_{\alpha_1\to\infty} M_{\alpha_1,\alpha_2,\dots,\alpha_n} = u(p_0) - v(p_0) \equiv L$$

allows us to find a large α_n^0 such that we have, in the notation of the lemma,

$$M_{\alpha_n^0} - L < \frac{\varepsilon}{2n}.$$

Note that, since we have a decreasing function, the absolute values are not needed and the inequality holds for all $\alpha_n > \alpha_n^0$. We may then find a large α_{n-1}^0 such that

$$M_{\alpha_{n-1}^0,\alpha_n^0} - M_{\alpha_n^0} < \frac{\varepsilon}{2n}.$$

Again, absolute values are not needed owing to the decreasing function, and the inequality holds for all larger α_{n-1}. We proceed iteratively until we have a large α_1^0 such that

$$M_{\alpha_1^0,\alpha_2^0,\ldots,\alpha_{n-1}^0,\alpha_n^0} - M_{\alpha_2^0,\ldots,\alpha_{n-1}^0,\alpha_n^0} < \frac{\varepsilon}{2n}.$$

By the triangle inequality, it follows that

$$M_{\alpha_1^0,\alpha_2^0,\ldots,\alpha_{n-1}^0,\alpha_n^0} - L < \varepsilon. \tag{4.11}$$

We then let each

$$\alpha_i > \max_j\{\alpha_j^0\}$$

and obtain

$$\begin{aligned} M_{\alpha_1,\alpha_2,\ldots,\alpha_{n-1},\alpha_n} - L &\leq M_{\alpha_1,\alpha_2,\ldots,\alpha_{n-1},\alpha_n^0} - L \\ &\leq M_{\alpha_1,\alpha_2,\ldots,\alpha_{n-1}^0,\alpha_n^0} - L \\ &\vdots \\ &\leq M_{\alpha_1^0,\alpha_2^0,\ldots,\alpha_{n-1}^0,\alpha_n^0} - L \end{aligned}$$

from the facts that each $\alpha_i > \alpha_i^0$ and $M_{\alpha_1,\alpha_2,\ldots,\alpha_{n-1},\alpha_n}$ is decreasing in each variable independently. The lemma then follows from (4.11). □

5. Comparison Principles

Comparison principles for general equations of the form $F = 0$ can be established using the previous section. In our first example, we consider strictly monotone elliptic functions F. That is, we require F to satisfy the following properties:

$$\sigma(r-s) \leq F(p,r,\eta,X) - F(p,s,\eta,X), \tag{5.1}$$

$$|F(p,r,\eta,X) - F(q,r,\eta,X)| \leq w_1(d_C(p,q)), \tag{5.2}$$

$$|F(p,r,\eta,X) - F(p,r,\eta,Y)| \leq w_2(\|Y-X\|), \tag{5.3}$$

$$|F(p,r,\eta,X) - F(p,r,\nu,X)| \leq w_3(|\|\eta\| - \|\nu\||), \tag{5.4}$$

where the constant $\sigma > 0$ and the functions $w_i \colon [0,\infty] \mapsto [0,\infty]$ satisfy $w_i(0^+) = 0$ for $i = 1,2,3$. We then formulate a comparison principle for such functions F.

THEOREM 5.1. *Let F satisfy* (5.1)–(5.4). *Let u be an upper semicontinuous subsolution and v a lower semicontinuous supersolution to*

$$F(p, f(p), \nabla_0 f(p), (D^2 f(p))^\star) = 0$$

in a domain Ω such that

$$\limsup_{q\to p} u(q) \leq \liminf_{q\to p} v(q)$$

when $p \in \partial\Omega$, *where both sides are neither* ∞ *nor* $-\infty$ *simultaneously. Then*

$$u(p) \le v(p)$$

for all $p \in \Omega$.

Proof. Suppose $\sup_\Omega(u - v) > 0$. Using the Grushin maximum principle from the previous section, we obtain

$$\begin{aligned}\sigma(u(p_\tau) - v(q_\tau)) &\le F(p_\tau, u(p_\tau), \tau\Upsilon_{p_\tau}, \mathcal{X}^\tau) - F(p_\tau, v(q_\tau), \tau\Upsilon_{p_\tau}, \mathcal{X}^\tau)\\ &= F(p_\tau, u(p_\tau), \tau\Upsilon_{p_\tau}, \mathcal{X}^\tau) - F(q_\tau, v(q_\tau), \tau\Upsilon_{q_\tau}, \mathcal{Y}^\tau)\\ &\quad + F(q_\tau, v(q_\tau), \tau\Upsilon_{q_\tau}, \mathcal{Y}^\tau) - F(p_\tau, v(q_\tau), \tau\Upsilon_{q_\tau}, \mathcal{Y}^\tau)\\ &\quad + F(p_\tau, v(q_\tau), \tau\Upsilon_{q_\tau}, \mathcal{Y}^\tau) - F(p_\tau, v(q_\tau), \tau\Upsilon_{p_\tau}, \mathcal{Y}^\tau)\\ &\quad + F(p_\tau, v(q_\tau), \tau\Upsilon_{p_\tau}, \mathcal{Y}^\tau) - F(p_\tau, v(q_\tau), \tau\Upsilon_{p_\tau}, \mathcal{X}^\tau).\end{aligned}$$

The first term is negative because u is a subsolution and v is a supersolution. Using (5.1)–(5.4) and Lemma 4.2 yields

$$0 < \sigma(u(p_\tau) - v(q_\tau)) \le w_1(d_C(p_\tau, q_\tau)) + w_2(\|R_\tau\|) + w_3(\tau|\|\Upsilon_{q_\tau}\| - \|\Upsilon_{p_\tau}\||),$$

which goes to 0 as τ approaches ∞. □

Our second example involves infinite harmonic functions. We wish to prove a comparison principle on a certain class of Grushin spaces using the multivariate maximum principle (Lemma 4.3). We will now use the notation $p_{\vec{\alpha}}$ to represent $p_{\alpha_1,\alpha_2,\ldots,\alpha_n}$, etc. We restrict ourselves to Grushin spaces in which the polynomials satisfy, for all i and j with $i < j$,

$$\lim_{\alpha_{i-1}\to\infty}\lim_{\alpha_{i-2}\to\infty}\cdots\lim_{\alpha_1\to\infty}\left(\frac{\partial\rho_j}{\partial x_i}\rho_i^2\rho_j\right)(p_{\vec{\alpha}}) - \left(\frac{\partial\rho_j}{\partial x_i}\rho_i^2\rho_j\right)(q_{\vec{\alpha}}) \sim (x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}}). \quad (5.5)$$

Note that (5.5) is trivially satisfied when $j = 1$, since then the expression is 0. When $j = 2$, we have $i = 1$ and the condition is satisfied because $\rho_2 = \rho_2(x_1)$. Observe that the case $n = 2$ then satisfies this condition, as well as arbitrary n with each ρ_j a function of x_1 only. We also observe that the case $n = 3$ with $\rho_3(x_1, x_2) = x_1 + x_2$ does not satisfy this condition, for

$$\left(\frac{\partial\rho_3}{\partial x_1}\rho_1^2\rho_3\right)(p_{\vec{\alpha}}) - \left(\frac{\partial\rho_3}{\partial x_1}\rho_1^2\rho_3\right)(q_{\vec{\alpha}}) = (x_1^{\vec{\alpha}} + x_2^{\vec{\alpha}}) - (y_1^{\vec{\alpha}} + y_2^{\vec{\alpha}}).$$

We thus have the following comparison principle.

THEOREM 5.2. *Let* G_n *be a Grushin space in which* (5.5) *holds for all* i, j *with* $i < j$. *Let* u *be an upper semicontinuous subsolution and* v *a lower semicontinuous supersolution to*

$$F_\varepsilon(\eta, X) = \min\{\|\eta\|^2 - \varepsilon^2, -\langle X\eta, \eta\rangle\} = 0$$

in a bounded domain Ω. *If*

$$\limsup_{q\to p} u(q) \le \liminf_{q\to p} v(q)$$

when $p \in \partial\Omega$, where both sides are neither ∞ nor $-\infty$ simultaneously, then

$$u(p) \leq v(p)$$

for all $p \in \Omega$.

Proof. We begin by noting that, as in [4], we can construct a strict supersolution of $F_\varepsilon = 0$ called w such that

$$F_\varepsilon(\nabla_0 w(q), (D^2 w(q))^\star) \geq \mu(q) \geq \mu > 0.$$

We may therefore assume without loss of generality that v is a strict supersolution associated with the constant μ. Because we are using the multivariate maximum principle (Lemma 4.3), we need only consider interior points by taking the α_j to be sufficiently large.

Proceeding as in Section 4, we have the vectors $\Upsilon_{p_{\vec{\alpha}}}$ and $\Upsilon_{q_{\vec{\alpha}}}$ defined by

$$\begin{aligned}(\Upsilon_{p_{\vec{\alpha}}})_i &= \rho_i(p_{\vec{\alpha}})\alpha_i(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}}),\\ (\Upsilon_{q_{\vec{\alpha}}})_i &= \rho_i(q_{\vec{\alpha}})\alpha_i(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}}).\end{aligned}$$

Using the construction of the vectors, we have

$$\begin{aligned}\|\Upsilon_{q_{\vec{\alpha}}}\|^2 - \|\Upsilon_{p_{\vec{\alpha}}}\|^2 &= \sum_{i=1}^{n} \alpha_i^2(\rho_i^2(q_{\vec{\alpha}}) - \rho_i^2(p_{\vec{\alpha}}))(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})^2\\ &= \sum_{i=2}^{n} \alpha_i^2(\rho_i^2(q_{\vec{\alpha}}) - \rho_i^2(p_{\vec{\alpha}}))(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})^2,\end{aligned}$$

since $\rho_1 \equiv 1$. We now note that every term in the sum lacks an α_1. Using that $\rho_i = \rho_i(x_1, x_2, \ldots, x_{i-1})$, we observe that

$$\begin{aligned}\lim_{\alpha_1\to\infty} \alpha_2^2(\rho_2^2(q_{\vec{\alpha}}) - \rho_2^2(p_{\vec{\alpha}}))(x_2^{\vec{\alpha}} - y_2^{\vec{\alpha}})^2 &= 0,\\ \lim_{\alpha_2\to\infty}\lim_{\alpha_1\to\infty} \alpha_3^2(\rho_3^2(q_{\vec{\alpha}}) - \rho_3^2(p_{\vec{\alpha}}))(x_3^{\vec{\alpha}} - y_3^{\vec{\alpha}})^2 &= 0,\\ &\vdots\\ \lim_{\alpha_{n-1}\to\infty}\lim_{\alpha_{n-2}\to\infty}\cdots\lim_{\alpha_1\to\infty} \alpha_n^2(\rho_n^2(q_{\vec{\alpha}}) - \rho_n^2(p_{\vec{\alpha}}))(x_n^{\vec{\alpha}} - y_n^{\vec{\alpha}})^2 &= 0.\end{aligned}$$

As a consequence, we are able to conclude that

$$\lim_{\alpha_n\to\infty}\lim_{\alpha_{n-1}\to\infty}\cdots\lim_{\alpha_2\to\infty}\lim_{\alpha_1\to\infty}\|\Upsilon_{q_{\vec{\alpha}}}\|^2 - \|\Upsilon_{p_{\vec{\alpha}}}\|^2 = 0. \tag{5.6}$$

Turning to the matrices $X^{\vec{\alpha}}$ and $Y^{\vec{\alpha}}$ in the second-order Euclidean jets, we construct the matrices $\mathcal{X}^{\vec{\alpha}}$ and $\mathcal{Y}^{\vec{\alpha}}$ by

$$\mathcal{X}_{ij}^{\vec{\alpha}} = \begin{cases} \rho_i(p_{\vec{\alpha}})\rho_j(p_{\vec{\alpha}})X_{ij}^{\vec{\alpha}} + \dfrac{1}{2}\dfrac{\partial\rho_j}{\partial x_i}(p_{\vec{\alpha}})\rho_i(p_{\vec{\alpha}})\alpha_j(x_j^{\vec{\alpha}} - y_j^{\vec{\alpha}}), & i \leq j,\\ \mathcal{X}_{ji}^{\vec{\alpha}}, & i > j,\end{cases}$$

and

$$\mathcal{Y}_{ij}^{\vec{\alpha}} = \begin{cases} \rho_i(q_{\vec{\alpha}})\rho_j(q_{\vec{\alpha}})Y_{ij}^{\vec{\alpha}} + \dfrac{1}{2}\dfrac{\partial\rho_j}{\partial x_i}(q_{\vec{\alpha}})\rho_i(q_{\vec{\alpha}})\alpha_j(x_j^{\vec{\alpha}} - y_j^{\vec{\alpha}}), & i \leq j, \\ \mathcal{Y}_{ji}^{\vec{\alpha}}, & i > j. \end{cases}$$

We then have

$$(\Upsilon_{p_{\vec{\alpha}}}, \mathcal{X}^{\vec{\alpha}}) \in \bar{J}^{2,+}u(p_{\vec{\alpha}}),$$
$$(\Upsilon_{q_{\vec{\alpha}}}, \mathcal{Y}^{\vec{\alpha}}) \in \bar{J}^{2,+}u(q_{\vec{\alpha}}).$$

The matrices $X^{\vec{\alpha}}$ and $Y^{\vec{\alpha}}$ satisfy a relation similar to the estimate (4.3)—namely, for vectors ε and χ we have

$$\langle X^{\vec{\alpha}}\varepsilon, \varepsilon\rangle_{\text{eucl}} - \langle Y^{\vec{\alpha}}\kappa, \kappa\rangle_{\text{eucl}} \leq \langle C\chi, \chi\rangle_{\text{eucl}},$$

where again the vector $\chi = (\varepsilon, \kappa)$. This time, however, the matrix C is a block matrix of the form

$$\begin{pmatrix} B & -B \\ -B & B \end{pmatrix}$$

whose submatrix B is defined by

$$B_{ij} = \begin{cases} \alpha_i + \sigma 2\alpha_i^2, & i = j, \\ 0, & i \neq j, \end{cases}$$

for a fixed small constant σ.

Using the construction of the matrices, we are now able to compute

$$\begin{aligned} &\langle \mathcal{X}^{\vec{\alpha}}\Upsilon_{p_{\vec{\alpha}}}, \Upsilon_{p_{\vec{\alpha}}}\rangle - \langle \mathcal{Y}^{\vec{\alpha}}\Upsilon_{q_{\vec{\alpha}}}, \Upsilon_{q_{\vec{\alpha}}}\rangle \\ &\quad \leq \langle B(\widetilde{\Upsilon_{p_{\vec{\alpha}}}} - \widetilde{\Upsilon_{q_{\vec{\alpha}}}}), \widetilde{\Upsilon_{p_{\vec{\alpha}}}} - \widetilde{\Upsilon_{q_{\vec{\alpha}}}}\rangle \\ &\qquad + \sum_{j=1}^{n}\sum_{i<j}\alpha_j(x_j^{\vec{\alpha}} - y_j^{\vec{\alpha}})\Bigg(\left(\frac{\partial\rho_j}{\partial x_i}\rho_i\right)(p_{\vec{\alpha}})(\Upsilon_{p_{\vec{\alpha}}})_i(\Upsilon_{p_{\vec{\alpha}}})_j \\ &\qquad\qquad - \left(\frac{\partial\rho_j}{\partial x_i}\rho_i\right)(q_{\vec{\alpha}})(\Upsilon_{q_{\vec{\alpha}}})_i(\Upsilon_{q_{\vec{\alpha}}})_j\Bigg). \end{aligned}$$

We recall (as in Section 4) that, given a vector κ, $\tilde{\kappa}$ is the Grushin twist of the original vector using the Main Lemma. Therefore, this sum can be expressed as

$$\begin{aligned} &\sum_{i=1}^{n}(\alpha_i + \sigma 2\alpha_i^2)(\rho_i^2(p_{\vec{\alpha}}) - \rho_i^2(q_{\vec{\alpha}}))^2\alpha_i^2(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})^2 \\ &\quad + \sum_{j=1}^{n}\sum_{i<j}\alpha_j^2\alpha_i(x_i^{\vec{\alpha}} - y_i^{\vec{\alpha}})(x_j^{\vec{\alpha}} - y_j^{\vec{\alpha}})^2\Bigg(\left(\frac{\partial\rho_j}{\partial x_i}\rho_i^2\rho_j\right)(p_{\vec{\alpha}}) - \left(\frac{\partial\rho_j}{\partial x_i}\rho_i^2\rho_j\right)(q_{\vec{\alpha}})\Bigg). \end{aligned}$$

Because $\rho_1 \equiv 1$, we see that the term corresponding to $i = 1$ in the first sum and the terms corresponding to $j = 1$ in the second sum are zero. Proceeding as in the vector difference estimate, we observe that the first sum has no α_1 terms and that the construction of the polynomials again produces

$$\lim_{\alpha_1\to\infty}(\alpha_2+\sigma 2\alpha_2^2)(\rho_2^2(p_{\vec{\alpha}})-\rho_2^2(q_{\vec{\alpha}}))^2\alpha_2^2(x_2^{\vec{\alpha}}-y_2^{\vec{\alpha}})^2=0,$$
$$\lim_{\alpha_2\to\infty}\lim_{\alpha_1\to\infty}(\alpha_3+\sigma 2\alpha_3^2)(\rho_3^2(p_{\vec{\alpha}})-\rho_3^2(q_{\vec{\alpha}}))^2\alpha_3^2(x_3^{\vec{\alpha}}-y_3^{\vec{\alpha}})^2=0,$$
$$\vdots$$
$$\lim_{\alpha_{n-1}\to\infty}\lim_{\alpha_{n-2}\to\infty}\cdots\lim_{\alpha_1\to\infty}(\alpha_n+\sigma 2\alpha_n^2)(\rho_n^2(p_{\vec{\alpha}})-\rho_n^2(q_{\vec{\alpha}}))^2\alpha_n^2(x_n^{\vec{\alpha}}-y_n^{\vec{\alpha}})^2=0;$$

we may therefore conclude that

$$\lim_{\alpha_n\to\infty}\lim_{\alpha_{n-1}\to\infty}\cdots\lim_{\alpha_1\to\infty}\sum_{i=1}^{n}(\alpha_i+\sigma 2\alpha_i^2)(\rho_i^2(p_{\vec{\alpha}})-\rho_i^2(q_{\vec{\alpha}}))^2\alpha_i^2(x_i^{\vec{\alpha}}-y_i^{\vec{\alpha}})^2=0.$$

We now turn to the second sum. First, consider the term where $j=2$ (which forces $i=1$). We note that

$$\left(\frac{\partial\rho_2}{\partial x_1}\rho_1^2\rho_2\right)(p_{\vec{\alpha}})-\left(\frac{\partial\rho_2}{\partial x_1}\rho_1^2\rho_2\right)(q_{\vec{\alpha}})\sim(x_1^{\vec{\alpha}}-y_1^{\vec{\alpha}})$$

and so we obtain

$$\lim_{\alpha_1\to\infty}\alpha_2^2\alpha_1(x_1^{\vec{\alpha}}-y_1^{\vec{\alpha}})(x_2^{\vec{\alpha}}-y_2^{\vec{\alpha}})^2\left(\left(\frac{\partial\rho_2}{\partial x_1}\rho_2\right)(p_{\vec{\alpha}})-\left(\frac{\partial\rho_2}{\partial x_1}\rho_2\right)(q_{\vec{\alpha}})\right)=0.$$

Next, consider the terms where $j>2$. We denote

$$T_{ij}=\alpha_j^2\alpha_i(x_i^{\vec{\alpha}}-y_i^{\vec{\alpha}})(x_j^{\vec{\alpha}}-y_j^{\vec{\alpha}})^2\left(\left(\frac{\partial\rho_j}{\partial x_i}\rho_i^2\rho_j\right)(p_{\vec{\alpha}})-\left(\frac{\partial\rho_j}{\partial x_i}\rho_i^2\rho_j\right)(q_{\vec{\alpha}})\right).$$

Since $i<j$, we can easily control

$$\lim_{\alpha_{i-1}\to\infty}\lim_{\alpha_{i-2}\to\infty}\cdots\lim_{\alpha_1\to\infty}T_{ij}$$

through the polynomials, since T_{ij} contains only α_i and α_j. In particular, after evaluating these limits, assumption (5.5) leaves us with

$$T_{ij}\sim\alpha_j^2\alpha_i(x_i^{\vec{\alpha}}-y_i^{\vec{\alpha}})(x_j^{\vec{\alpha}}-y_j^{\vec{\alpha}})^2(x_i^{\vec{\alpha}}-y_i^{\vec{\alpha}}),$$

so that taking $\alpha_i\to\infty$ produces 0. We then conclude that

$$\lim_{\alpha_n\to\infty}\lim_{\alpha_{n-1}\to\infty}\cdots\lim_{\alpha_2\to\infty}\lim_{\alpha_1\to\infty}\langle\mathcal{X}^{\vec{\alpha}}\Upsilon_{p_{\vec{\alpha}}},\Upsilon_{p_{\vec{\alpha}}}\rangle-\langle\mathcal{Y}^{\vec{\alpha}}\Upsilon_{q_{\vec{\alpha}}},\Upsilon_{q_{\vec{\alpha}}}\rangle=0. \tag{5.7}$$

Proceeding with the equation F_ε, we assume the maximum occurs at an interior point. Since we may reduce our discussion to interior points, we know that u is a viscosity subsolution at $p_{\vec{\alpha}}$ and that v is a viscosity supersolution at $q_{\vec{\alpha}}$. We then subtract the two equations to obtain

$$\begin{aligned}0<\mu&\le F_\varepsilon(\Upsilon_{q_{\vec{\alpha}}},\mathcal{Y}^{\vec{\alpha}})-F_\varepsilon(\Upsilon_{p_{\vec{\alpha}}},\mathcal{X}^{\vec{\alpha}})\\&=\max\{\|\Upsilon_{q_{\vec{\alpha}}}\|^2-\|\Upsilon_{p_{\vec{\alpha}}}\|^2,\langle\mathcal{X}^{\vec{\alpha}}\Upsilon_{p_{\vec{\alpha}}},\Upsilon_{p_{\vec{\alpha}}}\rangle-\langle\mathcal{Y}^{\vec{\alpha}}\Upsilon_{q_{\vec{\alpha}}},\Upsilon_{q_{\vec{\alpha}}}\rangle\}.\end{aligned}$$

We thus arrive at a contradiction via equations (5.6) and (5.7). □

Uniqueness of infinite harmonic functions then follows as in [4].

6. C^1_{sub} Absolute Minimizers are Viscosity Infinite Harmonic

Before proceeding, we recall the important derivatives. The horizontal gradient of the function u is defined as

$$\nabla_0 u = (X_1 u, X_2 u, \ldots, X_n u),$$

and the symmetrized horizontal second-order derivative matrix has entries

$$(D^2 u)^\star_{ij} = \frac{1}{2}(X_i X_j u + X_j X_i u).$$

These derivatives are used to define the infinite Laplacian by

$$\Delta_{0,\infty} u = \langle (D^2 u)^\star \nabla_0 u, \nabla_0 u \rangle.$$

Our proof of Theorem 1.5 is based on the following lemma, which is proved in [5] and is an extension of the result of Crandall [8]. The proof is included here in the interest of completeness.

LEMMA 6.1. *Let* $u \in \mathrm{Lip}(B(0,r))$, $u(0) = 0$, *and set* $\mu > 0$. *If for* $\varepsilon > 0$ *small enough one can find a function* $V_\varepsilon \in C^2_{\text{sub}}(\overline{U_\varepsilon})$, *where* $U_\varepsilon \subset B\big(0, \sqrt{\varepsilon/\mu}\big)$ *is a neighborhood of* 0, *such that*

$$\begin{aligned} V_\varepsilon(0) &= -\varepsilon, \\ V_\varepsilon &< u \ \text{ in } U_\varepsilon, \\ V_\varepsilon &= u \ \text{ on } \partial U_\varepsilon, \\ u &< V_\varepsilon \ \text{ outside the ball } B\big(0, \sqrt{\varepsilon/\mu}\big), \end{aligned} \tag{6.1}$$

and

$$|\nabla_0 V_\varepsilon| = 1 \ \text{ in a neighborhood of } U_\varepsilon, \tag{6.2}$$

then u cannot be absolutely minimizing in $B(0,r)$.

Proof. We argue by contradiction. Assume that u is an absolute minimizer; then

$$|\nabla_0 u| \le 1 \tag{6.3}$$

a.e. in U_ε. Let γ be the horizontal curve obtained as a solution of the ODE

$$\frac{d}{dt}\gamma = -\Big[(\nabla_0 V_\varepsilon)\nabla_0\Big]\Big|_\gamma \quad \text{and} \quad \gamma(0) = 0. \tag{6.4}$$

Since $|(d/dt)\gamma| = 1$ and since γ is horizontal, there exist $\varepsilon > 0$ and $C_\varepsilon > 0$ such that, for $0 \le t < \varepsilon$,

$$C_\varepsilon^{-1} t \le |\gamma|_{G_n} \le C_\varepsilon t. \tag{6.5}$$

Since $u \in C^1_{\text{sub}}(B(0,r))$, we compute

$$\frac{d}{dt}(V_\varepsilon(\gamma) - u(\gamma)) = -\langle \nabla_0 V_\varepsilon, \nabla_0 V_\varepsilon(\gamma) - \nabla_0 u(\gamma)\rangle. \tag{6.6}$$

From (6.3) it follows that

$$\frac{d}{dt}(V_\varepsilon(\gamma) - u(\gamma)) < 0.$$

Hence, for any $t > 0$,

$$(V_\varepsilon(\gamma) - u(\gamma))(t) \le (V_\varepsilon(\gamma) - u(\gamma))(0) = -\varepsilon. \tag{6.7}$$

The latter implies that $\gamma \in U_\varepsilon$ for any $t > 0$, but this is in contradiction with (6.5) and with the fact that $U_\varepsilon \subset B\big(0, \sqrt{\varepsilon/\mu}\big)$. In fact, the curve γ will exit the ball $B\big(0, \sqrt{\varepsilon/\mu}\big)$ after a time roughly equivalent to $\sqrt{\varepsilon}$. □

The proof of Theorem 1.5 is now reduced to finding a function V_ε as in Lemma 6.1. We begin by assuming that the C^1_{sub} function $u \in \mathrm{Lip}(\Omega)$ is an absolute minimizer and fails to be ∞-harmonic in the viscosity sense at the point p_0. We first consider the case when $p_0 = 0$. Without loss of generality, we assume that u is not a viscosity subsolution and so there is a function ϕ such that

$$\begin{aligned} \phi(0) &= 0, \\ \phi(p) &> u(p) \ \text{ for } p \text{ near } 0, \\ \Delta_{0,\infty}\phi(0) &< 0. \end{aligned}$$

Next, we denote the derivatives of ϕ by

$$\begin{aligned} D &\equiv \nabla_0 \phi(0), \\ H &\equiv (D^2\phi(0))^\star, \\ B &\equiv -HD, \end{aligned}$$

and define the antisymmetric matrix C by

$$C_{ij} = \frac{1}{2}[X_j, X_i]\phi(0).$$

First, we wish to construct a symmetric matrix M such that

$$M > H \quad \text{and} \quad MD = CD.$$

As in Section 2, we define the set $\mathcal{N}$ by

$$j \in \mathcal{N} \iff \rho_j^0 = 0.$$

We begin by defining the $n \times n$ symmetric matrix P by the formula

$$Pz = z + \frac{\langle (C-H)D, z\rangle}{\langle (C-H)D, D\rangle}(C-H)D - \frac{\langle D, z\rangle}{\langle D, D\rangle} D$$

for any $z = (z_1, z_2, \ldots, z_n)$. It is easy to see that

$$\begin{aligned} \langle Pz, z\rangle &> 0 \quad \text{for } z \neq 0, \\ \langle Pz, z'\rangle &= \langle Pz', z\rangle, \\ PD &= (C-H)D. \end{aligned}$$

Thus, the matrix $\mathcal{M} = P + H$ has the properties that

$$\mathcal{M} > H \quad \text{and} \quad \mathcal{M}D = CD.$$

We then define the matrix M by

$$M_{ij} = \begin{cases} H_{ij}, & i \in \mathcal{N} \text{ or } j \in \mathcal{N}, \\ \mathcal{M}_{ij}, & i, j \notin \mathcal{N}. \end{cases}$$

To show that $M > H$, we observe that

$$\sum_{i,j=1}^{n} (M - H)_{ij}\alpha_i\alpha_j = \sum_{i,j \notin \mathcal{N}} (\mathcal{M} - H)_{ij}\alpha_i\alpha_j,$$

which is positive because $\mathcal{M} > H$. We next observe that if $i \in \mathcal{N}$ then

$$X_i X_j \phi(0) = 0,$$

and so

$$\sum_{j=1}^{n} M_{ij} D_j = \sum_{j=1}^{n} H_{ij} D_j = \frac{1}{2}\sum_{j=1}^{n} X_j X_i \phi(0) D_j = \sum_{j=1}^{n} C_{ij} D_j.$$

If $i \notin \mathcal{N}$, then invoking the properties of the matrix $\mathcal{M}$ and recalling that $D_j = 0$ if $j \in \mathcal{N}$ allows us to write

$$\sum_{j=1}^{n} M_{ij} D_j = \sum_{j \notin \mathcal{N}} M_{ij} D_j = \sum_{j \notin \mathcal{N}} \mathcal{M}_{ij} D_j = \sum_{j \notin \mathcal{N}} C_{ij} D_j = \sum_{j=1}^{n} C_{ij} D_j. \tag{6.8}$$

Once we have the matrix M, we define the hyperplane L by

$$L = \left\{ \sum_{i \notin \mathcal{N}} x_i X_i : \sum_{j \notin \mathcal{N}} B_j x_j = 0 \right\}.$$

Note that $D \notin L$, since $B \cdot D > 0$, and that $D_j = 0$ when $j \in \mathcal{N}$. We have k degrees of freedom for the indexes in $\mathcal{N}$ and $n - k - 1$ degrees of freedom outside of $\mathcal{N}$, so this is indeed a hyperplane.

Next, we define a C^2_{sub} function (using the Taylor polynomial from Section 2) as follows:

$$\begin{aligned} h(x) = {} & \sum_{j \notin \mathcal{N}} \frac{1}{\rho_j^0} x_j D_j + \frac{1}{2} \sum_{j \notin \mathcal{N}} \frac{1}{(\rho_j^0)^2} (x_j)^2 M_{jj} \\ & + \sum_{\substack{i,j \notin \mathcal{N} \\ i \le j}} x_i x_j \left(\frac{1}{\rho_i^0 \rho_j^0} M_{ij} - \frac{1}{2} \frac{1}{(\rho_j^0)^2} \frac{\partial \rho_j}{\partial x_i}(0) D_j \right) \\ & + \sum_{m=1}^{k} x_{i_m} \frac{\partial}{\partial x_{i_m}} \phi(0); \end{aligned}$$

we then solve the Cauchy problem

$$\begin{cases} |\nabla V_\varepsilon| = 1 \ \text{ in a neighborhood } O \text{ of } 0, \\ V_\varepsilon = h_\varepsilon \ \text{ on } L \cap O, \end{cases} \tag{6.9}$$

subject to the constraint

$$\partial_{x_i} V(0) = \partial_{x_i}\phi(0), \quad i = 1, 2, \dots, n. \tag{6.10}$$

In order to obtain a solution, we need to verify that L is not characteristic at $p_0 = 0$. Define the functions

$$f(p, \zeta) = \frac{1}{2}\sum_{i=1}^{n} (\rho_i(x_1, x_2, \dots, x_{i-1})\zeta_i)^2$$

and

$$\psi(p) = \sum_{j \notin \mathcal{N}} B_j x_j.$$

We compute Euclidean derivatives and use equation (6.10) to obtain:

$$\begin{aligned} \partial_{\zeta_i} f(0, \nabla_{\text{eucl}} V(0)) &= (\rho_i^0)^2 \frac{\partial}{\partial x_i} V(0) \\ &= (\rho_i^0) X_i \phi(0); \\ \partial_{x_i}\psi(0) &= \begin{cases} B_i, & i \notin \mathcal{N}, \\ 0, & i \in \mathcal{N}. \end{cases} \end{aligned}$$

Hence, the Euclidean inner product can be computed as

$$\langle \nabla_\zeta f(0, \nabla_{\text{eucl}} V(0)), \nabla_p \psi(0) \rangle_{\text{eucl}} = \sum_{i \notin \mathcal{N}} X_i \phi(0) B_i = B \cdot D > 0.$$

By [15], it follows that L is not characteristic.

Having verified that a solution V exists, we want to show that

$$\frac{1}{2}(X_i X_j + X_j X_i) V(0) \stackrel{\text{def}}{=} \mathcal{V}_{ij} = M_{ij}.$$

Differentiating (6.9) with respect to X_i $(i \notin \mathcal{N})$ produces

$$\sum_{j=1}^{n} (X_i X_j V) X_j V = 0.$$

When we evaluate at zero, equation (6.8) yields

$$\begin{aligned} \mathcal{V}_{ij} D_j &= \sum_{j=1}^{n} \frac{1}{2}(X_i X_j V(0) + X_j X_i V(0)) D_j \\ &= \sum_{j=1}^{n} \frac{1}{2}[X_j, X_i] V(0) D_j = \sum_{j=1}^{n} C_{ij} D_j \\ &= \sum_{j=1}^{n} M_{ij} D_j. \end{aligned}$$

On the other hand, if $i \in \mathcal{N}$ then

$$\begin{aligned}
\mathcal{V}_{ij} D_j &= \frac{1}{2}(X_j X_i V(0)) D_j \\
&= \frac{1}{2}\rho_j^0 \frac{\partial \rho_i}{\partial x_j}(0) \frac{\partial}{\partial x_i} V(0) D_j \\
&= \frac{1}{2}\rho_j^0 \frac{\partial \rho_i}{\partial x_j}(0) \frac{\partial}{\partial x_i} \phi(0) D_j \\
&= \frac{1}{2}(X_j X_i \phi(0)) D_j = H_{ij} D_j = M_{ij} D_j.
\end{aligned}$$

Thus, off of L we have $\mathcal{V} \equiv M$. Since $V = h$ on L, we know from the computation of the Taylor polynomial in Section 2 that

$$\frac{1}{2}(X_i X_j + X_j X_i) h(0) = M_{ij}.$$

An explicit calculation then shows that, for any vectors w_1 and w_2 in L,

$$\langle M w_1, w_2 \rangle = \langle \mathcal{V} w_1, w_2 \rangle.$$

Using this equality and polarization identities, we conclude that

$$\mathcal{V} \equiv M.$$

We have thus constructed a function V satisfying

$$\begin{aligned}
V(0) &= \phi(0) = 0, \\
\nabla_0 V(0) &= D, \\
\mathcal{V} &= M > H.
\end{aligned}$$

We define the function V_ε by $V_\varepsilon = V - \varepsilon$ for some $\varepsilon > 0$.

From the general Taylor theorem (see [10]) we immediately obtain that

$$V_\varepsilon > \phi - \varepsilon$$

in a neighborhood of the origin. For a possibly smaller neighborhood and for a small $\mu > 0$, we have

$$V_\varepsilon(p) > \phi(p) - \varepsilon + \mu \sum_{i=1}^{n} x_i^2.$$

Observe that for $\varepsilon > 0$ small enough and for $r \geq \sqrt{\varepsilon/\mu}$ with $p \in \partial B(0, r)$,

$$u(p) \leq \phi(p) \leq \phi(p) + (\mu r^2 - \varepsilon) = \phi(p) - \varepsilon + \mu \sum_{i=1}^{n} x_i^2 < V_\varepsilon(p). \quad (6.11)$$

Hence there exists a neighborhood U_ε of the origin (we may assume without loss of generality that it is connected) such that

$$V_\varepsilon = u \text{ on } \partial U_\varepsilon \quad \text{and} \quad V_\varepsilon < u \text{ in } U_\varepsilon. \quad (6.12)$$

We then have $U_\varepsilon \subset B\big(0, \sqrt{\varepsilon/\mu}\,\big)$. The contradiction now stems from (6.11), (6.12), and Lemma 6.1.

Next, suppose that the function u fails to be a viscosity solution at some point p_0. We then perform a change of variables by replacing x_i with $x_i + x_i^0$. Then, for each vector field X_i, we have the vector field

$$Y_i = \rho_i(x_1 + x_1^0, x_2 + x_2^0, \ldots, x_{i-1} + x_{i-1}^0)\frac{\partial}{\partial x_i}$$

for $i = 2, 3, \ldots, n$ and $Y_1 = X_1$. Then, elementary calculus shows that equation (1.1) produces

$$[Y_i, Y_j] = [X_i, X_j]$$

for $i < j$. Because

$$Y_i(0) = 0 \iff X_i(p_0) = 0,$$

we conclude that the Y_i vector fields vanish at the same order at 0 as the corresponding X_i vector fields vanish at p_0. In addition, we also define $\tilde{u}(p) = u(p + p_0)$ so that $\mathrm{Lip}(\tilde{u}) = \mathrm{Lip}(u)$ and $\tilde{u}$ is not a viscosity solution at the origin. We then can use the proof of Lemma 6.1 to obtain the desired contradiction for $\tilde{u}$ at the origin, completing the proof of Theorem 1.5. □

Having proved Theorem 1.5, it is desirable to remove the regularity assumption. Since dilations do not exist in general and since mollifiers may not possess the necessary technical properties even in the special cases when dilations do exist, the removal of the regularity assumption using this technique is still an open question. C. Y. Wang [19] recently announced related results in his preprint. By appealing to the Euclidean case, he answers a more general question.

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